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Relativistic Holographic Hydrodynamics

from Black Hole Horizons

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Abstract

We consider the AdS/CFT correspondence in the hydrodynamic regime up to the second order in a derivative expansion. We demonstrate that the fluid conservation equations are equivalent to Einstein's constraint equations projected on different hyper-surfaces. We derive that result for hyper-surfaces of the form $r = R(x^\alpha)$ up to the first order in a derivative expansion of the metric. At the second order expansion, we introduce the notion of different black hole horizons, and focus on two particular horizon hyper-surfaces: the event horizon and the apparent horizon. We calculate the temperature and entropy current for the apparent horizon and show that the latter agrees with the area increase theorem for the black hole, and differs from the entropy current calculated for the event horizon.

Contents

Chapter 1

Introduction

The AdS/CFT correspondence proposed in [1], plays a significant role in understanding strongly coupled conformal gauge field theories. One important application of this duality is the study of the effective description of these strongly coupled gauge field theories in the long wavelength regime[1]. This effective description is the hydrodynamic plasma description. The correspondence states that a strongly coupled conformal field theory corresponds to a weakly coupled string theory, and the latter, in a certain regime, reduces to the Einstein classical field equations with a negative cosmological constant in an asymptotically Anti de Sitter space-time. This correspondence is suitable for describing out-of equilibrium fluid flow (the hydrodynamic description) which is dual to out-of-equilibrium long wavelength dynamics of a black brane. An example for this is the fluid-gravity correspondence presented in [2]. The AdS/CFT duality has a remarkable application in describing strongly coupled QCD type plasma [3].

The equations governing relativistic hydrodynamics are conservation equations, namely the relativistic navier-stokes equations [4, 5]. Those equations can be found from parts of Einstein's equations

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[1]We are actually using the small Knudsen number regime, which is explained in the following chapters.

in the dual gravitational description [6, 7]. In this thesis we will project those equations on different hyper-surfaces and verify that every hyper-surface reproduce the same conservation equations.

From thermodynamics we get the notion of many thermodynamic quantities that can be generalized for hydrodynamics, for example: temperature, chemical potential and entropy current. The latter will be dual to the area of a hyper-surface that admits the area increase theorem. In this thesis we will explore the different hyper-surfaces and we will focus on two of them, namely: the event and apparent horizons.

We will start by describing the relativistic hydrodynamic fluid and the derivative expansion in chapter 2. At the end of the chapter we will explain the Weyl formalism, which we will work with in the rest of the thesis. In chapter 3 we will set the stage for the gravity description and in chapter 4 we will provide basic geometric quantities. At the end of chapter 4 we will present parts of Einstein's equation which are dual to the conservation equations of the hydrodynamic fluid, and in the following two chapter 5, 6 we will explore those equations for different hyper-surfaces. For this we will introduce the apparent horizon hyper-surface in chapter 6. In chapter 7 we will calculate the entropy current and the temperature for the apparent horizon, and we will present our conclusion in chapter 8.

Chapter 2

Relativistic Hydrodynamics

Hydrodynamics [5] is a theoretical model which describes the behavior of fluids in motion. It treats the fluid as having local domains which are in equilibrium, hence one can define in those domains definite thermodynamic quantities such as temperature, pressure, entropy, etc.

One can address a fluid as a relativistic only if the macroscopic velocity of the fluid is close to the speed of light.

2.1 Hydrodynamics equations

In order to derive the relativistic equations of fluid dynamics we first need to derive the form of the energy momentum *stress tensor*.

If we look on a d dimensional fluid element[1] in its *local rest frame,* we will realize that in that frame Pascal law is valid, which means that the pressures on the different surfaces surrounding it are equal. Therefore, we get $T^{ii} = p$. On that local frame the component T^{00} of the stress tensor is just the local internal energy density of the fluid ε. The other components of the stress tensor are zero $T^{0i} = 0$.

If we introduce the d-velocity of the fluid u^{μ}, then in the local rest frame of the fluid we get $u^0 = -1$ and $u^i = 0$. Then the energy

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[1] Can be understood as a volume within the fluid which contains a large number of molecules

momentum stress tensor of an ideal fluid will be:

$$T^{\mu\nu}_{Ideal} = \varepsilon u^{\mu} u^{\nu} + p\left(\eta^{\mu\nu} + u^{\mu} u^{\nu}\right) \tag{2.1}$$

If the fluid equation of state is provided, then one can find the connection between the pressure and the energy density.

From the definition of the d-velocity in the local rest frame, we can calculate its norm (which will be valid in any frame):

$$u^{\mu} u_{\mu} = -1 \tag{2.2}$$

We can define the tensor which multiplies the pressure on (2.1) as the *projector tensor* $P^{\mu\nu}$, because it projects along the transverse direction of the fluid velocity.

We would like to consider a fluid which undergoes viscosity effects, so we will need to introduce a viscous stress tensor $\tau^{\mu\nu}$, which will be added to the ideal stress tensor:

$$T^{\mu\nu} = T^{\mu\nu}_{Ideal} + \tau^{\mu\nu} \tag{2.3}$$

We have some freedom in the determination of the viscous stress tensor that we would like to set. First, we will consider the viscous stress tensor to be proportional to the derivatives of $u^{\mu}(x)$. Second, we will need to consider how viscosity will affect the fluid in its local rest frame. In order to do so we will require that the energy density and the momentum densities in the local rest frame will not change due to viscous effects, which means $\tau^{00} = 0$, $\tau^{0i} = 0$, and because the fluid velocity $u^{i} = 0$, we have in the local rest frame of the fluid the following expression:

$$\tau^{\mu\nu} u_{\nu} = 0 \tag{2.4}$$

This result is correct not only in the rest frame of the fluid but also in any Lorentz frame, this result is called the *Landau—Lifshitz*

frame.

We will also specify the equations of motion that can be written in a simple form, because they are just the conservation equations of the stress tensor:

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.5}$$

2.2 Hydrodynamics as an effective Conformal Field Theory

When working in the regime of small Knudsen number (long wavelength) the relativistic field theory has a relativistic hydrodynamic description [4].

The Knudsen number is just the correlation length of the fluid ℓ_{cor} divided by the characteristic length scale L of the variations of the macroscopic fields, so we can write the condition of effective hydrodynamics description as:

$$Kn = \frac{\ell_{cor}}{L} \ll 1 \tag{2.6}$$

Under this condition we can expand the stress tensor in a small parameter $Kn \ll 1$ and get:

$$T^{\mu\nu}(x) = \sum_{l=0}^{\infty} T_l^{\mu\nu}(x), \qquad T_l^{\mu\nu} \sim (Kn)^l \tag{2.7}$$

where $T_l^{\mu\nu}(x)$ is determined locally by the value of the velocity $u^\mu(x)$, the pressure $p(x)^2$ and their derivatives. For example the zeroth order will be the ideal with no derivatives.

If we consider a relativistic Conformal Field Theory with a finite temperature T, we get:

$$T_\mu^\mu = 0 \tag{2.8}$$

---

[2]We are assuming that the equation of state has been given.

Therefore, we can find the equation of state from this condition and we get $p = \frac{\varepsilon}{d-1}$. From dimensional analysis we have $p = aT^d$ and $\varepsilon = (d-1)\,aT^d$ where a is a normalization coefficient. Then equation (2.1) takes the form:

$$T^{\mu\nu}_{Ideal} = aT^d \left(\eta^{\mu\nu} + du^\mu u^\nu \right) \tag{2.9}$$

In order to find the viscous stress tensor $\tau^{\mu\nu}$, we will use the conformality requirement (2.8), from which we will see that the viscous stress tensor has to be traceless, i.e., $\tau^\mu_\mu = 0$. In addition, if we apply the equations of motion (2.5) to the zeroth order stress tensor, we get a connection between the derivatives of the velocity of the fluid $u^\mu(x)$ to the derivatives of the fluid temperature $T(x)$, so we can replace, in the next order, the derivatives of $T(x)$ with the derivatives of $u^\mu(x)$. We can follow this procedure to all higher order in this iterative form, in order to eliminate completely the derivatives of $T(x)$.

By those two conditions and the symmetry of the stress tensor, one can find the stress tensor by taking all possible terms with different coefficients[8]. Here we will present the complete stress tensor to second order in derivatives expansion,

$$T^{\mu\nu} = aT^d \left(\eta^{\mu\nu} + du^\mu u^\nu \right) - 2\eta\sigma^{\mu\nu} + \eta\tau_\Pi \Sigma^{\mu\nu}_{(0)} + \lambda_1 \Sigma^{\mu\nu}_{(1)} + \lambda_2 \Sigma^{\mu\nu}_{(2)} + \lambda_3 \Sigma^{\mu\nu}_{(3)} \tag{2.10}$$

where the shear tensor is defined by:

$$\sigma_{\mu\nu} = P^\alpha_\mu P^\beta_\nu \partial_{(\alpha} u_{\beta)} - \frac{\partial_\alpha u^\alpha}{d-1} P_{\mu\nu} \tag{2.11}$$

the vorticity tensor is defined by:

$$\omega_{\mu\nu} = P^\alpha_\mu P^\beta_\nu \partial_{[\alpha} u_{\beta]} \tag{2.12}$$

and

$$\Sigma^{\mu\nu}_{(0)} = 2P^{\mu\alpha}P^{\nu\beta}u^\lambda\partial_\lambda\sigma_{\alpha\beta}+2\frac{\partial_\alpha u^\alpha}{d-1}\sigma^{\mu\nu}, \quad \Sigma^{\mu\nu}_{(1)} = 4\sigma^\mu_\lambda\sigma^{\nu\lambda}-4\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P^{\mu\nu},$$

$$\Sigma^{\mu\nu}_{(2)} = 2\sigma^{\mu\lambda}\omega^\nu_\lambda + 2\sigma^{\nu\lambda}\omega^\mu_\lambda, \quad \Sigma^{\mu\nu}_{(3)} = \omega^\mu_\lambda\omega^{\nu\lambda} - \frac{\omega_{\alpha\beta}\omega^{\alpha\beta}}{d-1}P^{\mu\nu}. \quad (2.13)$$

The brackets represents symmetric or anti-symmetric tensors $A_{(\alpha\beta)} = \frac{1}{2}\left(A_{\alpha\beta} + A_{\beta\alpha}\right), \quad A_{[\alpha\beta]} = \frac{1}{2}\left(A_{\alpha\beta} - A_{\beta\alpha}\right).$

The coefficients $\eta, \tau_\Pi, \lambda_1, \lambda_2, \lambda_3$ are called *transport coefficients*, and in the literature [5] η is also called the *shear viscosity* coefficient.

2.3   Weyl covariant formulation

In this section, we will introduce the Weyl covariant formulation as done in [9], which is useful for conformal fluid.

We will consider a $d > 3$ conformal fluid in a curved background with metric $g_{\mu\nu}$[3] and we will look at some observables of the fluid. We will consider a conformal transformation:

$$g_{\mu\nu} = e^{2\phi}\widetilde{g}_{\mu\nu}, \quad g^{\mu\nu} = e^{-2\phi}\widetilde{g}^{\mu\nu} \qquad (2.14)$$

where $\phi = \phi\left(x^\mu\right)$.

As in the previous section we will denote by u^μ the d-velocity of the fluid with the normalization (2.2). From this normalization and (2.14) we can find the transformation rule of the d-velocity by: $g_{\mu\nu}u^\mu u^\nu = \widetilde{g}_{\mu\nu}\widetilde{u}^\mu\widetilde{u}^\nu = -1$ and we get: $u^\mu = e^{-\phi}\widetilde{u}^\mu$.

However, if we look on the transformation rule of the covariant derivative of u^μ we get:

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[3]In the rest of the thesis we will consider a flat background metric $\eta_{\mu\nu}$, thus for a flat background one needs to replace $g_{\mu\nu} \to \eta_{\mu\nu}$ and $\nabla_\lambda \to \partial_\lambda$.

$$\nabla_\mu u^\nu = e^{-\phi} \left(\widetilde{\nabla}_\mu \widetilde{u}^\nu + \delta^\nu_\mu \widetilde{u}^\sigma \partial_\sigma \phi - \widetilde{g}_{\mu\lambda} \widetilde{u}^\lambda \widetilde{g}^{\nu\sigma} \partial_\sigma \phi \right) \qquad (2.15)$$

which does not transform homogeneously. In order to deal with homogeneous transformations, which are allowed in a conformal fluid, we define a *Weyl covariant derivative* D_λ in the following manner: if a tensorial quantity $Q^{\mu\cdots}_{\nu\cdots}$ transforms $Q^{\mu\cdots}_{\nu\cdots} = e^{-\omega\phi}\widetilde{Q}^{\mu\cdots}_{\nu\cdots}$ then the weyl covariant derivative will transform: $D_\lambda Q^{\mu\cdots}_{\nu\cdots} = e^{-\omega\phi}\widetilde{D}_\lambda\widetilde{Q}^{\mu\cdots}_{\nu\cdots}$, where:

$$
\begin{aligned}
D_\lambda Q^{\mu\cdots}_{\nu\cdots} \equiv\ & \nabla_\lambda Q^{\mu\cdots}_{\nu\cdots} + \omega A_\lambda Q^{\mu\cdots}_{\nu\cdots} \\
& + [g_{\lambda\alpha}A^\mu - \delta^\mu_\lambda A_\alpha - \delta^\mu_\alpha A_\lambda]\, Q^{\alpha\cdots}_{\nu\cdots} + \ldots \qquad (2.16) \\
& - [g_{\lambda\nu}a^\alpha - \delta^\alpha_\lambda A_\nu - \delta^\alpha_\nu A_\lambda]\, Q^{\mu\cdots}_{\alpha\cdots} - \ldots
\end{aligned}
$$

Where A_λ is a one-form that can be determined uniquely by the requirements that the Weyl covariant derivative of the d-velocity will be traceless and transverse to the fluid direction, i.e., $D_\mu u^\mu = 0$ and $u^\lambda D_\lambda u^\mu = 0$. We get:

$$A_\nu = u^\lambda \nabla_\lambda u_\nu - \frac{\nabla_\lambda u^\lambda}{d-1} u_\nu \qquad (2.17)$$

Note that the Weyl covariant derivative is metric compatible, i.e: $D_\lambda g_{\mu\nu} = 0$.

We will present here additional transformations of some observables of the fluid:

$$
\begin{aligned}
D_\mu u^\nu &= \nabla_\mu u^\nu + u_\mu u^\lambda \nabla_\lambda u^\nu - \frac{\nabla_\lambda u^\lambda}{d-1} P_\mu^\nu \\
&= \sigma_\mu^\nu + \omega_\mu^\nu = e^{-\phi} \widetilde{D}_\mu \widetilde{u}^\nu, \qquad\qquad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
\sigma^{\mu\nu} &\equiv \frac{1}{2} \left(P^{\mu\lambda} \nabla_\lambda u^\nu + P^{\nu\lambda} \nabla_\lambda u^\mu \right) - \frac{\nabla_\lambda u^\lambda}{d-1} P_\mu^\nu \\
&= \frac{1}{2} \left(D^\mu u^\nu + D^\nu u^\mu \right) = e^{-3\phi} \widetilde{\sigma}^{\mu\nu}, \qquad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
\omega^{\mu\nu} &\equiv \frac{1}{2} \left(P^{\mu\lambda} \nabla_\lambda u^\nu - P^{\nu\lambda} \nabla_\lambda u^\mu \right) \\
&= \frac{1}{2} \left(D^\mu u^\nu - D^\nu u^\mu \right) = e^{-3\phi} \widetilde{\omega}^{\mu\nu} \qquad (2.20)
\end{aligned}
$$

$$
D_\lambda \sigma^\lambda_{\ \mu} = \left(\nabla_\lambda - (d-1) A_\lambda \right) \sigma^\lambda_{\ \mu} \qquad (2.21)
$$

$$
D_\lambda \omega^\lambda_{\ \mu} = \left(\nabla_\lambda - (d-3) A_\lambda \right) \omega^\lambda_{\ \mu} \qquad (2.22)
$$

$$
u^\lambda D_\lambda \sigma_{\mu\nu} = P_\mu^{\ \alpha} P_\nu^{\ \beta} \nabla_\lambda \sigma_{\alpha\beta} + \frac{\nabla_\lambda u^\lambda}{d-1} \sigma_{\mu\nu} \qquad (2.23)
$$

We will also define a *Weyl covariant Riemann curvature tensor* for a vector field with a weight ω , i.e. $V^\mu = e^{-\omega\phi} \widetilde{V}^\mu$ by

$$
[D_\mu, D_\nu] V_\lambda = \omega \mathcal{F}_{\mu\nu} V_\lambda + \mathcal{R}_{\mu\nu\lambda}^{\ \ \ \alpha} V_\alpha \qquad (2.24)
$$

with

$$
\mathcal{F}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \qquad (2.25)
$$

$$
\mathcal{R}_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} + \mathcal{F}_{\mu\nu} g_{\lambda\sigma} - 4 \delta^\alpha_{[\mu} g_{\nu][\lambda} \delta^\beta_{\sigma]} \left(\nabla_\alpha A_\beta + A_\alpha A_\beta - \frac{A_\alpha A^\alpha}{2} g_{\alpha\beta} \right)
$$
$$
(2.26)
$$

After defining the Weyl covariant Riemann curvature tensor, it

is natural to define the *Weyl-covariantized Ricci tensor* $\mathcal{R}_{\mu\nu}$ and *Weyl-covariantized Ricci scalar* $\mathcal{R}$ by,

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + (d-2)\left(\nabla_\mu A_\nu + A_\mu A_\nu - A_\alpha A^\alpha g_{\mu\nu}\right) + g_{\mu\nu}\nabla_\lambda A^\lambda + \mathcal{F}_{\mu\nu} \tag{2.27}$$

$$\mathcal{R} = R + 2\left(d-1\right)\nabla_\lambda A^\lambda - \left(d-2\right)\left(d-1\right) A_\lambda A^\lambda \tag{2.28}$$

Note that we will work with a flat metric on the boundary, hence, the Riemann curvature tensor, the Ricci tensor, and the Ricci scalar of the boundary are zero (However, their Weyl definitions are not zero).

In order to evaluate fluid dynamics to the second order, we will need to take all sorts of terms with two Weyl covariant derivatives. This work was done in [10] and can also be found in [9] which we use for our notations.

Chapter 3

Fluid Gravity Correspondence

The fluid gravity correspondence is the AdS/CFT or gauge/gravity duality in a long wavelength regime. This duality relates a particular strongly coupled non-abelian gauge theory in d dimensions to string theory, which in a certain regime reduces to classical gravity in $d+1$ dimensions. The regimes we will work with are the planar limit in the field theory and the long wavelength regime. The latter is a small Knudsen number regime, and the CFT can be described effectively by a hydrodynamic description. Hence, we have a duality between a d-dimensional relativistic fluid to a classical gravity in $d+1$ dimensions which is just Einstein's equations with a negative cosmological constant in an asymptotically Anti de Sitter space-time. We can describe this duality holographically and relate to the fluid as "living on the boundary" of the whole $d+1$ space-time, which we will refer to as the bulk. Moreover, the small Knudsen number suggests an expansion of the fields, from both side of the duality, in a derivative expansion. This will allow us to solve Einstein's equations order by order. It turns out that parts of Einstein's equations at a certain order implement the stress tensor conservation equations of the fluid at a lower order. This statement will be checked in the following chapters, but not before we will give the basic set up of

the bulk space-time geometry in this chapter. Two reviews on the gauge/gravity duality can be found in [11, 12].

3.1 Preliminaries: Schwarzschild black holes in AdS_{d+1}

We would like to look at the dual description of fluid from the gravity perspective. In order to do so, we will have to solve Einstein's equations with a negative cosmological constant with a particular choice of units $(R_{AdS} = 1)$ and we get:

$$
\begin{aligned}
E_{ab} &= R_{ab} - \tfrac{1}{2}Rg_{ab} - \tfrac{d(d-1)}{2}g_{ab} = T_{ab}^{matter} = 0 \\
&\implies R_{ab} + dg_{ab} = 0, \quad R = -d\,(d+1)
\end{aligned}
\tag{3.1}
$$

One solution to these equations is just the AdS_{d+1} solution which is dual to a vacuum state in the CFT. Another class of solutions is described by the "boosted Schwarzschild black branes",

$$
ds^2 = -2u_\mu dx^\mu dr - r^2 f\,(br)\,u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu
\tag{3.2}
$$

with,

$$
f\,(r) = 1 - \frac{1}{r^d}, \quad u^\mu = \left(\gamma, \gamma u^i\right), \quad \gamma = \frac{1}{\sqrt{1 - (u^i)^2}}
$$

$$
b = \frac{d}{4\pi T}, \quad \left(u^i\right)^2 = u^i u_i
\tag{3.3}
$$

This solution is dual to a CFT at a finite temperature T, where the velocity u^i and the temperature are constants.

The metrics (3.2) describe the uniform black brane in an asymptotically Anti de Sitter space-time written in the ingoing Eddington-
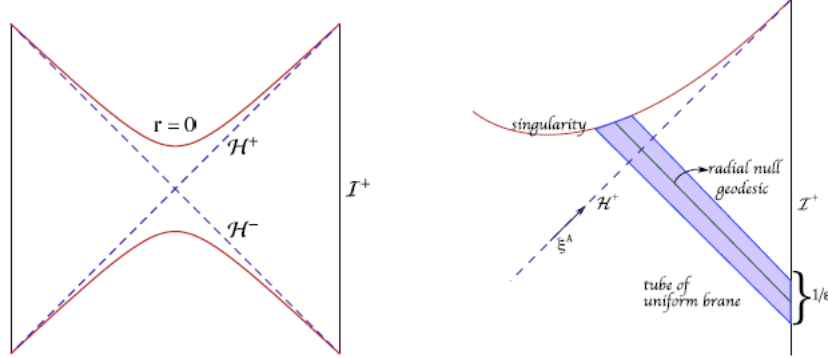
Figure 3.1: Penrose diagram of the uniform black brane and the causal structure of the space-times dual to fluid mechanics illustrating the tube structure. The dashed line in the second figure denotes the future event horizon, while the shaded tube indicates the region of space-time over which the solution is well approximated by a tube of the uniform black brane. (Taken from [11]).

Finkelstein coordinates[1], at temperature T, moving at velocity u^i.

The sets of solutions characterized by d parameters give us d possible different solutions. The parameters are just the temperature and the d parameters which define the d-velocity, so we have $d+1$ parameters, and because of the normalization (2.2) they are reduced to d parameters.

If we had written the solution in Schwarzschild type coordinates[2] we could easily have read from the metric the location of the event horizon $r = \frac{1}{b}$. One can look on the left image of figure 3.1 in order to understand in a pictorial way the casual structure of space-time.

---

[1] These coordinates exhibit a regular solution at the event horizon for all orders of expansion.

[2] The unboosted solution in Schwarzschild type coordinates

$$ds^2 = -r^2 f(br) dt^2 + \frac{dr^2}{r^2 f(br)} + r^2 \delta_{ij} dx^i dx^j$$

with δ_{ij} is Kronecker's delta. Note that it is not exhibits regularity at the event horizon location.

3.2 Away from global equilibrium

Now we would like to consider an out-of-equilibrium black brane in such manner that the dual description will give us hydrodynamics of a viscous fluid [2]. In order to do so, we promote the parameters to be dependent on the boundary coordinates x^μ, but not on the radial direction r, and we get from (3.2):

$$ds^2 = -2u_\mu\left(x^\alpha\right)dx^\mu dr - r^2 f\left(b\left(x^\alpha\right)r\right)u_\mu\left(x^\alpha\right)u_\nu\left(x^\alpha\right)dx^\mu dx^\nu + r^2 P_{\mu\nu}\left(x^\alpha\right)dx^\mu dx^\nu$$
$$(3.4)$$

However (3.4), denoted as $g^{(0)}\left(u\left(x^\alpha\right),b\left(x^\alpha\right)\right)$[3], is not a solution of Einstein's equations.

The procedure to satisfy Einstein's equations goes as follow:

1. Expand the velocity and temperature fields in a small parameter.

2. Use the ansatz so that after a while the system relaxes and go back to the static solution (3.2).

3. Calculate the metric order-by-order to satisfy Einstein's equations.

We will do the expansions in the manner of effective field theory. As we have done in (2.7), we will take all possible derivatives along the boundary coordinates (i.e., x^μ) to get the expansions,

$$u^\mu\left(x^\alpha\right) = \sum_{l=0}^{\infty} u_{(l)}^\mu\left(x^\alpha\right), \quad b\left(x^\alpha\right) = \sum_{l=0}^{\infty} b_{(l)}\left(x^\alpha\right) \qquad (3.5)$$

It is possible to prove that every derivative will come with a factor of b, and if the fields are changing slowly when looking on a large length scale L, we have that every term in the expansions is

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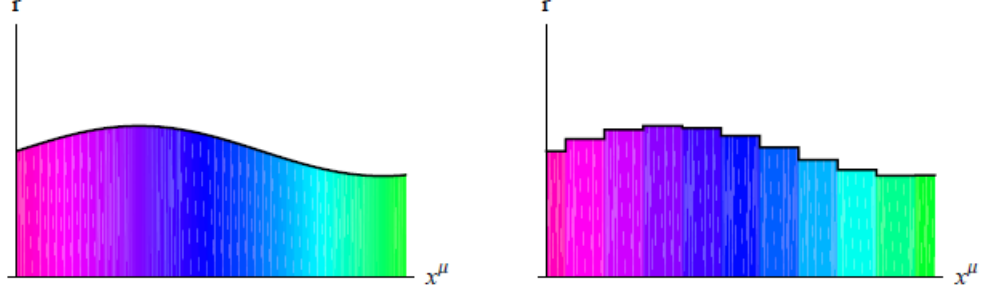[3]Consider the fields as the zeroth order of 3.5

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Figure 3.2: Cartoon of 'tubewise approximation' of slowly-varying configuration by a corresponding piecewise-constant one. (Taken from [12]).

proportional to $\left(\frac{b}{L}\right)^l \sim \left(\frac{1}{TL}\right)^l$. In other words, in order that these expansions be justified we will require the Knudsen number to be small 2.6.

This regime of expansions can be visualized as tubes that go along the radial direction and their basis is on the boundary (figure 3.1 right picture). In those tubes we can define a definite velocity and definite temperature (figure 3.2).

Now we will plug the expansions (3.5) into the metric and get $g_{ab} = \sum_{l=0}^{\infty} g_{ab}^{(l)}$. Then we will require that this metric solves Einstein's equation and we can determine the metric order-by-order in an iterative form.

The Einstein's equations can be written in the form:

$$\mathbb{H}\left[g^{(0)}\left(b_{(0)}, u_{(0)}^{\mu}\right)\right] g_{ab}^{(n)}\left(x^{\alpha}\right) = s_n \qquad (3.6)$$

where $\mathbb{H}$ is a linear second order differential operator in the r variable alone. Note that it does not depend on n, so that it is the same operator in any order of the expansion. Moreover, the precise form of this operator at the point x^{μ} depends only on the values of $b_{(0)}$ and $u_{(0)}^{\mu}$ at x^{μ} but not on the derivatives of these functions at that point. The source term s_n is a regular source term which contains

complicated combinations of derivatives of the fields and the fields themselves and it is different in any order of expansion.

It turns out that it is possible to classify Einstein's equations into two categories:

1. the *constraint equations* $E_\mu^r = 0$.

2. the *dynamical equations* $E_r^r = 0$, $E_\mu^\nu = 0$.

The d constraint equations are defined as the equations that are of first order in r derivatives. It follows that they are connecting the derivatives of the temperature field with the derivatives of the velocity field. Hence we can use them in order to eliminate the temperature derivatives in all order of expansion. The remaining $\frac{d(d+1)}{2}$ dynamical equations are determined by the next order metric[4]. Then all together we have $\frac{(d+1)(d+2)}{2}$ which is the number of equations that (3.6) possess.

3.3 The Bulk Metric

In this section we will present the general form of metrics that can describe the bulk geometry of AdS and are also in accord with the conformal fluid on the boundary of the AdS space-time. Then we will present the second order solution of the metric which satisfy (3.1) in a Weyl covariant formulation [13].

We start by writing the most general metric that satisfies the gauge choice of the previous section:

$$ds^2 = -2u_\mu\left(x^\alpha\right) dx^\mu \left(dr + \mathcal{V}_\nu\left(r, x^\alpha\right) dx^\nu\right) + \mathfrak{G}_{\mu\nu}\left(r, x^\alpha\right) dx^\mu dx^\nu,$$

(3.7)

---

[4]There is a redundancy among the remaining equations which leaves $\frac{d(d-1)}{2}$ independent 'dynamical' equations, this freedom is eliminating by choosing the following gauge condition:

$$g_{rr} = 0, \quad g_{r\mu} = -u_\mu.$$

where $\mathfrak{G}_{\mu\nu}$ is transverse to the fluid velocity, i.e., $u^{\mu}\mathfrak{G}_{\mu\nu} = 0$.

We will also present the inverse of the bulk metric,

$$
\begin{aligned}
&u^{\mu}\left[\left(\partial_{\mu} - \mathcal{V}_{\mu}\partial_{r}\right) \otimes \partial_{r} + \partial_{r} \otimes \left(\partial_{\mu} - \mathcal{V}_{\mu}\partial_{r}\right)\right] \\
&+ \left(\mathfrak{G}^{-1}\right)^{\mu\nu}\left(\partial_{\mu} - \mathcal{V}_{\mu}\partial_{r}\right) \otimes \left(\partial_{\nu} - \mathcal{V}_{\nu}\partial_{r}\right)
\end{aligned}
\tag{3.8}
$$

or in another form,

$$
ds^2 = 2\left(u^{\mu} - \left(\mathfrak{G}^{-1}\right)^{\mu\nu}\mathcal{V}_{\nu}\right)\partial_{\mu}\otimes\partial_r + \left(\left(\mathfrak{G}^{-1}\right)^{\mu\nu}\mathcal{V}_{\mu}\mathcal{V}_{\nu} - 2u^{\mu}\mathcal{V}_{\mu}\right)\partial_r\otimes\partial_r + \left(\vartheta^{-1}\right)^{\mu\nu}\partial_{\mu}\otimes\partial_{\nu}
\tag{3.9}
$$

where the symmetric tensor $\left(\mathfrak{G}^{-1}\right)^{\mu\nu}$ is uniquely defined by the relations: $u_{\mu}\left(\mathfrak{G}^{-1}\right)^{\mu\nu} = 0$ and $\left(\mathfrak{G}^{-1}\right)^{\mu\lambda}\mathfrak{G}_{\lambda\nu} = P_{\nu}^{\mu}$.

We will now write the solution to the second order,

$$
\begin{aligned}
\mathcal{V}_{\mu} = {}& rA_{\mu} + \frac{1}{d-2}\left[D_{\lambda}\omega^{\lambda}_{\mu} - D_{\lambda}\sigma^{\lambda}_{\mu} + \frac{\mathcal{R}}{2(d-1)}u_{\mu}\right] - \frac{2L(br)}{(br)^{d-2}}P_{\mu}^{\nu}D_{\lambda}\sigma^{\lambda}_{\nu} \\
& -\frac{u_{\mu}}{2(br)^{d}}\left[r^2\left(1 - (br)^{d}\right) - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} - (br)^2 K_2(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{G}_{\mu\nu} = {}& r^2 P_{\mu\nu} - \omega^{\lambda}_{\mu}\omega_{\lambda\nu} \\
& +2(br)^2 F(br)\left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma^{\lambda}_{\mu}\sigma_{\lambda\nu}\right] - 2(br)^2 K_1(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} \\
& -2(br)^2 H_1(br)\left[u^{\lambda}D_{\lambda}\sigma_{\mu\nu} + \sigma^{\lambda}_{\mu}\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu}\right] \\
& +2(br)^2 H_2(br)\left[u^{\lambda}D_{\lambda}\sigma_{\mu\nu} + \omega^{\lambda}_{\mu}\sigma_{\lambda\nu} + \omega^{\lambda}_{\nu}\sigma_{\mu\lambda}\right]
\end{aligned}
$$

and,

17

$$\left(\mathfrak{G}^{-1}\right)^{\mu\nu} = \frac{1}{r^2}P^{\mu\nu} + \frac{1}{r^4}\omega^{\mu\lambda}\omega_\lambda{}^\nu$$
$$-\frac{2b^2}{r^2}F\left(br\right)\left[\frac{1}{b}\sigma^{\mu\nu} - F\left(br\right)\sigma^\mu_\lambda\sigma^{\lambda\nu}\right] + \frac{2b^2}{r^2}K_1\left(br\right)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P^{\mu\nu}$$
$$+\frac{2b^2}{r^2}H_1\left(br\right)\left[u^\lambda D_\lambda\sigma^{\mu\nu} + \sigma^\mu_\lambda\sigma^{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P^{\mu\nu}\right]$$
$$-\frac{2b^2}{r^2}H_2\left(br\right)\left[u^\lambda D_\lambda\sigma^{\mu\nu} + \omega^\mu_\lambda\sigma^{\lambda\nu} + \omega^\nu_\lambda\sigma^{\mu\lambda}\right]$$

with,

$$F\left(br\right) \equiv \int_{br}^{\infty}\frac{y^{d-1}-1}{y\left(y^d-1\right)}dy$$

$$H_1\left(br\right) \equiv \int_{br}^{\infty}\frac{y^{d-2}-1}{y\left(y^d-1\right)}dy$$

$$H_2\left(br\right) \equiv \frac{1}{2}F\left(br\right)^2 - \int_{br}^{\infty}\frac{d\xi}{\xi\left(\xi^d-1\right)}\int_1^\xi\frac{y^{d-2}-1}{y\left(y^d-1\right)}dy$$

$$K_1\left(br\right) \equiv \int_{br}^{\infty}\frac{d\xi}{\xi^2}\int_\xi^\infty dy\,y^2F'\left(y\right)^2$$

$$K_2\left(br\right) \equiv \int_{br}^{\infty}\frac{d\xi}{\xi^2}\left[1 - \xi\left(\xi-1\right)F'\left(\xi\right) - 2\left(d-1\right)\xi^{d-1}\right.$$
$$\left.+ \left(2\left(d-1\right)\xi^d - \left(d-2\right)\right)\int_\xi^\infty dy\,y^2F'\left(y\right)^2\right]$$

$$L\left(br\right) \equiv \int_{br}^{\infty}\xi^{d-1}d\xi\int_\xi^\infty dy\,\frac{y-1}{y^3\left(y^d-1\right)}$$

Chapter 4

Horizon Dynamics

Given a manifold of space-time one would like to investigate the geodesic dynamics equations of that manifold. To do that one has to solve the Einstein equations. However, it is possible to consider slices of that manifold, for instance hyper-surfaces. Einstein's equations projected onto a hyper-surface take a simpler form called the Gauss-Codazzi equations.

In order to understand the geodesic dynamics on hyper-surfaces of the space-time geometry (Gauss-Codazzi equations), we need to introduce some basic quantities that characterize the hyper-surfaces [14] (induced metric, extrinsic curvature, etc.).

4.1 Geometric Preliminaries

A *hyper-surface* is a sub-manifold that can either be space-like, time-like, or null[1] (we will almost not deal with the latter in this section). A particular hyper-surface $\mathcal{S}$ is selected either by imposing a restriction on the coordinates, $\Phi(x^a) = 0$, or by providing a parametric equation of the form $x^a = x^a(y^\mu)$.

The normal vector to the surface $\mathcal{S}$ is defined by $\Phi_{,a}$, because the value of Φ changes only in the direction orthogonal to $\mathcal{S}$.

---

[1] For the null case refer to [15, 16]

A *unit normal* $\hat{m}^a$ is defined by:

$$\hat{m}_a = \frac{\Phi_{,a}}{\sqrt{|g^{ab}\Phi_{,a}\Phi_{,b}|}} \tag{4.1}$$

where we demand that $\hat{m}^a$ point in the direction of increasing Φ: $\hat{m}^a\Phi_{,a} > 0$, and we get its norm to be:

$$\hat{m}^a\hat{m}_a = \begin{cases} -1 & if\ S\ is\ space - like \\ +1 & if\ S\ is\ time - like \end{cases} \tag{4.2}$$

In the case of a null surface we can only define a class of null normal co-vectors, in the following way:

$$\ell_a = f\,\Phi_{,a} \tag{4.3}$$

where f is an arbitrary function. The norm of the class of vectors is:

$$\ell^a\ell_a = 0 \tag{4.4}$$

.

Now we would like to define a metric that would be intrinsic to the hyper-surface. The *induced metric* $h_{\mu\nu}$ to a hyper-surface $\mathcal{S}$ is obtained by restricting the line element to displacements confined to the hyper-surface. Using the parametric equations $x^a = x^a\left(y^\mu\right)$, we find that the vectors,

$$e_\mu^a = \frac{\partial x^a}{\partial y^\mu} \tag{4.5}$$

are tangent to curves contained in $\mathcal{S}$. (This means that $e_\mu^a\hat{m}_a = 0$[2]). Therefore, we now write the restriction of the line element for displacements within $\mathcal{S}$:

---

[2] $e_\mu^a\hat{m}_a = 0 \propto \frac{\partial x^a}{\partial y^\mu}\frac{\partial \Phi}{\partial x^a} = \frac{\partial \Phi}{\partial y^\mu} = 0$. It is zero because the function is constant in the directions of the surface.

$$
\begin{aligned}
ds^2 &= g_{ab}dx^a dx^b \\
&= g_{ab}\left(\frac{\partial x^a}{\partial y^\mu}dy^\mu\right)\left(\frac{\partial x^b}{\partial y^\nu}dy^\nu\right) \\
&= h_{\mu\nu}dy^\mu dy^\nu
\end{aligned}
\tag{4.6}
$$

where

$$
h_{\mu\nu} = g_{ab}e_\mu^a e_\nu^b
\tag{4.7}
$$

Note that the induced metric is a scalar under bulk space-time coordinate transformations, $x^a \rightarrow x^{a'}$. However, it transforms as a tensor under the hyper-surface coordinate transformations, $y^\mu \rightarrow y^{\mu'}$.

With the aid of the induced metric we can write the decomposition of the bulk metric:

$$
g^{ab} = h^{ab} \pm \hat{m}^a \hat{m}^b
\tag{4.8}
$$

with:

$$
h^{ab} = h^{\alpha\beta}e_\alpha^a e_\beta^b
\tag{4.9}
$$

where the plus sign in (4.8) represents the time-like hyper-surfaces, and the minus represents the space-like hyper-surfaces.

This decomposition can be defined to every hyper-surface that is not null.

We now would like to understand how tangent tensor fields are differentiated. First we define such fields:

A *tangent tensor field* $A^{ab\cdots}$ is a tensor field that is defined only on $\mathcal{S}$ and is purely tangent to the hyper-surface. Such fields admit the following decomposition:

$$A^{ab\cdots} = A^{\mu\nu\cdots}e^a_\mu e^b_\nu \cdots \tag{4.10}$$

Equation (4.10) implies that $A^{ab\cdots}\hat{m}_a = A^{ab\cdots}\hat{m}_b = \cdots = 0$ which confirm that $A^{ab\cdots}$ is tangent to the hyper-surface. The tangent tensor $A^{\mu\nu}$ indices are lowered and raised with the induced metric on the hyper-surface $h_{\mu\nu}$ and $h^{\mu\nu}$.

Now we can define an *intrinsic covariant derivative*. We will use a tangent vector field for simplicity: $A^a = A^\mu e^a_\mu$, $A^a\hat{m}_a = 0$, $A_\mu = A_a e^a_\mu$. The intrinsic covariant derivative of A_μ is defined by the projection of $A_{a;b}$ onto the hyper-surface:

$$A_{\mu|\nu} \equiv A_{a;b}e^a_\mu e^b_\nu \tag{4.11}$$

It is possible to prove that equation (4.11) can take the form of,

$$A_{\mu|\nu} \equiv A_{\mu,\nu} - \Gamma^\sigma_{\mu\nu}A_\sigma \tag{4.12}$$

where

$$\Gamma^\sigma_{\mu\nu} = h^{\sigma\lambda}\left(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu.\sigma}\right) \tag{4.13}$$

We can now introduce the *extrinsic curvature* of a hyper-surface, which is just the projection of the covariant derivative of the normal to the hyper-surface onto the hyper-surface S:

$$K_{\mu\nu} \equiv \hat{m}_{a;b}e^a_\mu e^b_\nu \tag{4.14}$$

This quantity is a symmetric tensor and it tells us how the hyper-surface S is embedded in the bulk. We can see this by taking its trace: $K = h^{\mu\nu}K_{\mu\nu} = \hat{m}^a_{;a}$ (we used the decomposition equation (4.8)) which is the expansion[3] of a congruence of geodesics that intersect the hyper-surface S orthogonally. When $K > 0$ (the con-

---

[3]This term will be explained in chapter 6.

gruence is diverging), the hyper-surface is *convex* and if $K < 0$ (the congruence is converging), the hyper-surface is *concave.*

4.2 Gauss-Codazzi equations

We are ready to introduce the Einstein equations on a hyper-surface S. We start by defining a purely intrinsic curvature tensor by the relation:

$$A^{\sigma}_{|\mu\nu} - A^{\sigma}_{|\nu\mu} = R^{\sigma}_{\lambda\nu\mu} A^{\lambda} \tag{4.15}$$

where the explicit expression for the purely intrinsic curvature tensor is:

$$R^{\sigma}_{\lambda\nu\mu} = \Gamma^{\sigma}_{\lambda\mu,\nu} - \Gamma^{\sigma}_{\lambda\nu,\mu} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\lambda\mu} - \Gamma^{\sigma}_{\alpha\mu}\Gamma^{\alpha}_{\lambda\nu} \tag{4.16}$$

We now want to establish a connection between the intrinsic curvature tensor to the Riemann curvature tensor defined in the bulk geometry of space-time. It can be proven [14] that the relation is:

$$R^{d}_{abc} e^{a}_{\alpha} e^{b}_{\beta} e^{c}_{\gamma} = R^{\sigma}_{\alpha\beta\gamma} e^{d}_{\sigma} \pm \left(K_{\alpha\beta|\gamma} - K_{\alpha\gamma|\beta} \right) \hat{m}^{d} \pm K_{\alpha\beta} \hat{m}^{d}_{;c} e^{c}_{\gamma} \mp K_{\alpha\gamma} \hat{m}^{d}_{;b} e^{b}_{\beta} \tag{4.17}$$

where again the plus sign represents the time-like hyper-surfaces and the minus represents the space-like hyper-surfaces.

Projecting along m_d and using the symmetry of Riemann curvature tensor[4] gives:

$$R_{adcb} \hat{m}^{d} e^{a}_{\alpha} e^{b}_{\beta} e^{c}_{\gamma} = \left(K_{\alpha\beta|\gamma} - K_{\alpha\gamma|\beta} \right) \tag{4.18}$$

We will now contract the indices by using the metric $h^{\alpha\gamma}$ with equation (4.18). Then, by using equation (4.8) and the symmetry of Riemann curvature tensor, we get:

---

[4] $R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$

$$R_{ab}\hat{m}^a e_\beta^b = \left(K_{\beta|\alpha}^\alpha - K_{,\beta} \right) \tag{4.19}$$

If we look at the bulk metric g_{ab} contracted with $\hat{m}^a e_\beta^b$ we get,

$$g_{ab}\hat{m}^a e_\beta^b = \hat{m}_b e_\beta^b = 0 \tag{4.20}$$

This means that we can replace the left hand side of (4.19) with $(R_{ab} + Cg_{ab})\,\hat{m}^a e_\beta^b$, where C is an arbitrary coefficient, because every term that is proportional to the metric will vanish. For instance, we can replace the left hand side with the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ to get the *Gauss-Codazzi equations,*

$$G_{ab}\hat{m}^a e_\beta^b = K_{\beta|\alpha}^\alpha - K_{,\beta} \tag{4.21}$$

Because we are solving the Einstein equations with a cosmological constant we would like to replace the left hand side of (4.19) with E_{ab}, which we introduced in equation (3.1), and we get,

$$E_{ab}\hat{m}^a e_\beta^b = K_{\beta|\alpha}^\alpha - K_{,\beta} \tag{4.22}$$

As mentioned in the equations (4.21) and (4.22) we are eventually reducing these equations, because of (4.20), to (4.19) and these are the equations we have to solve in order to find the geodesic dynamic equations.

4.3 The Constraint equations

In the previous chapter we stated and showed that Einstein's equations can be divided into two categories, the dynamical equations and the constraint equations. In this section we will deal with the latter.

<u>Claim</u>: The constraint equations $E_\mu^r = 0$ with the metric (3.7), which solves the dynamic Einstein's equation, can be written in the

form:

$$E_{ab}\xi^b e^a_\mu = 0 \tag{4.23}$$

where ξ^b is normal to any hyper-surface S (time-like, space-like and null) that is defined by $\Phi = r - r(x^\alpha) = 0$. Thus ξ^b is defined by $\xi^b = f\, g^{ba}\Phi_{,a}$, where f is an arbitrary normalization function.

<u>Proof</u>: Let us set $f = 1$ for convenience reasons and look at the LHS of (4.23):

$$
\begin{aligned}
E_{ab}\xi^b e^a_\mu &= R_{ab}\xi^b e^a_\mu \\
&= e^a_\mu R_{ab} g^{bc}\partial_c\Phi \\
&= e^a_\mu R^r_a - e^a_\mu R^\nu_a \partial_\nu r(x^\alpha) \\
&= R^r_\mu + R^r_r \partial_\mu r(x^\alpha) - R^\nu_\mu \partial_\nu r(x^\alpha) - R^\nu_r \partial_\mu r(x^\alpha)\partial_\nu r(x^\alpha) \\
&= R^r_\mu - d\partial_\mu r(x^\alpha) + d\delta^\nu_\mu \partial_\nu r(x^\alpha) - (g^{\nu\sigma}R_{\sigma r} + g^{\nu r}R_{rr})\partial_\mu r(x^\alpha)\partial_\nu r(x^\alpha) \\
&= R^r_\mu + d g^{\nu\sigma} g_{\sigma r}\partial_\mu r(x^\alpha)\partial_\nu r(x^\alpha) \\
&= R^r_\mu - d g^{\nu\sigma} u_\sigma \partial_\mu r(x^\alpha)\partial_\nu r(x^\alpha) \\
&= R^r_\mu
\end{aligned}
$$

where we got the fifth line by using Einstein's equations (3.1) and we got the sixth line by using the relation: $g^{\mu\nu}g_{r\nu} = -\mathfrak{G}^{\mu\nu}u_\nu = 0$. Using the relation: $E^r_\mu = R^r_\mu$ completes the proof.

∎

For the non-null case we can take $\xi^a = \hat{m}^a$[5] in (4.23) and we get from (4.22) the Gauss-Codazzi equations with a vanishing matter stress tensor:

$$K^\nu_{\mu|\nu} - K_{,\mu} = 0 \tag{4.24}$$

---

[5]Note that the normalization function f can be different from one.

If we look at the constraint equations on the boundary with the normal vector to the boundary surface $\xi^a = g^{ar}$, then the constraint equations, according to [17], take the simple form of the equations of motion for the boundary fluid (2.5).

The constraint equations just connect the derivatives of the fluid d-velocity with the derivatives of the temperature (or the inverse temperature b), and of course the d-velocity and the temperature fields depend, by our definition (3.5), on the coordinates perpendicular to the r coordinate, which are the boundary coordinates x^α. For this reason we can claim that the constraint equations are the same for every hyper-surface $\Phi = r - r(x^\alpha)$ of space-time. In the following chapters we check this claim explicitly.

Chapter 5

The Second Order Constraint Equations

5.1 Hyper-Surfaces At Constant Radial Location

In this section we will check the constraint equations to the first order in derivative expansion for a constant radial coordinate hyper-surface, i.e., $\Phi = r - R$, where R is constant.

For reasons of convenience, we will write here only the zeroth and first order parts of the metric (3.7)

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\
&\quad + 2r^2 bF(br)\sigma_{\mu\nu} dx^\mu dx^\nu + \frac{2}{d-1} r u_\mu u_\nu \partial_\lambda u^\lambda dx^\mu dx^\nu - r u^\lambda \partial_\lambda (u_\mu u_\nu) dx^\mu dx^\nu
\end{aligned}
\tag{5.1}
$$

The components of the inverse metric to the first order are:

$$
g^{rr} = r^2 f(br) - \frac{2}{d-1} r \partial_\lambda u^\lambda, \ \ g^{r\mu} = u^\mu - \frac{1}{r} u^\lambda \partial_\lambda u^\mu, \ \ g^{\mu\nu} = \frac{1}{r^2} P^{\mu\nu} - \frac{2bF(br)}{r^2}\sigma^{\mu\nu}
\tag{5.2}
$$

Our basis vectors (4.5) are:

$$
e^r_\mu = \frac{\partial R}{\partial x^\mu} = 0, \quad e^\nu_\mu = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu
\tag{5.3}
$$

The normal vector to this surface is:

$$\hat{m}^a = \frac{g^{ab}\partial_b\Phi}{\sqrt{|g^{ab}\partial_a\Phi\partial_b\Phi|}} = \frac{1}{\sqrt{|g^{rr}|}}g^{ar} \qquad (5.4)$$

The components of the one-form dual to the normal vector are:

$$\hat{m}_r = \frac{1}{\sqrt{|g^{rr}|}}, \quad \hat{m}_\mu = 0 \qquad (5.5)$$

In order that the last expression will be defined, we demand $g^{rr} \neq 0$. The surface Φ is time-like if $g^{rr} > 0$, or space-like if $g^{rr} < 0$.

Our goal is to write the constraint equations by using (4.24), so we will need to find the extrinsic curvature of the hyper-surface. We will calculate the mixed upper and lower indices of the extrinsic curvature in the following way:

$$
\begin{aligned}
K^\mu_\nu &= h^{\mu\sigma}K_{\sigma\nu} \\
&= h^{\mu\sigma}\hat{m}_{a;b}e^a_\sigma e^b_\nu \\
&= h^{\mu a}\hat{m}_{a;b}e^b_\nu \\
&= \left(g^{\mu a} \mp \hat{m}^\mu\hat{m}^a\right)\hat{m}_{a;b}e^b_\nu \qquad (5.6)\\
&= g^{\mu a}\hat{m}_{a;b}e^b_\nu \mp \hat{m}^\mu\hat{m}^a\hat{m}_{a;b}e^b_\nu \\
&= \hat{m}^\mu_{;b}e^b_\nu \\
&= \nabla_\nu\hat{m}^\mu
\end{aligned}
$$

where we used (5.3) to get the fourth line and $\hat{m}^a\hat{m}_{a;b} = 0$ to get the sixth line. After applying the covariant derivative, we get:

$$K^\mu_\nu = \nabla_\nu\hat{m}^\mu = \partial_\nu\hat{m}^\mu + \Gamma^\mu_{\nu a}\hat{m}^a = \partial_\nu\hat{m}^\mu + \Gamma^\mu_{\nu r}\hat{m}^r + \Gamma^\mu_{\nu\sigma}\hat{m}^\sigma \qquad (5.7)$$

In order to compute the constraint equations up to the second order in derivative expansion, we have to find the constraint equa-

tions that come from the zeroth order metric, and then apply them to the calculation of the constraint equations that comes from the first order metric. We show here the explicit form of the extrinsic curvature to the zeroth order,

$$K_\nu^{\mu(0)} = \cdots = \frac{1}{\sqrt{|f(br)|}} \left(-\frac{1}{2} \frac{d}{(br)^d} u^\mu u_\nu + f(br) \eta_\nu^\mu \right) \tag{5.8}$$

and its trace is,

$$K^{(0)} = \frac{1}{\sqrt{|f(br)|}} \left(\frac{1}{2} \frac{d}{(br)^d} + df(br) \right) \tag{5.9}$$

Inserting (5.8) and (5.9) to the Gauss-Codazzi equations (4.24) we get:

$$
\begin{aligned}
K_{\nu|\mu}^{\mu(0)} - K_{,\nu}^{(0)} &= \frac{d}{2(bR)^d \sqrt{|f(bR)|}} \left(2du_\nu u^\mu \partial_\mu \ln b - u^\mu \partial_\mu u_\nu \right. \\
&\qquad -u_\nu \partial_\mu u^\mu + (d+1) \partial_\nu \ln b \\
&\qquad \left. -dP_\nu^\mu \partial_\mu \ln b \right) = 0 \tag{5.10}
\end{aligned}
$$

Projecting (5.10) along u^ν we get:

$$
\begin{aligned}
\frac{d}{2(bR)^d \sqrt{|f(bR)|}} \left(\partial_\mu u^\mu - (d-1) u^\nu \partial_\nu \ln b \right) &= 0 \Rightarrow \\
\Rightarrow \partial_\mu u^\mu - (d-1) u^\nu \partial_\nu \ln b &= 0 \tag{5.11}
\end{aligned}
$$

Projecting (5.10) along P_σ^ν we get:

$$
\begin{aligned}
-\frac{d}{2(bR)^d \sqrt{|f(bR)|}} \left(u^\mu \partial_\mu u_\sigma - P_\sigma^\nu \partial_\nu \ln b \right) &= 0 \Rightarrow \\
\Rightarrow u^\mu \partial_\mu u_\sigma - P_\sigma^\nu \partial_\nu \ln b &= 0 \tag{5.12}
\end{aligned}
$$

Now we compute the extrinsic curvature to the first order from the first order bulk metric (5.1),

$$
\begin{aligned}
K_\nu^{\mu(0+1)} &= \cdots \\
&= \frac{1}{r\sqrt{|f\,(br)|}}\left(-\frac{1}{2}r\frac{d}{(br)^d}u^\mu u_\nu + rf\,(br)\,\eta_\nu^\mu + \sigma_\nu^\mu\right. \\
&\qquad\qquad +r^2 f\,(br)\,\partial_r F\,(br)\,b\sigma_\nu^\mu \\
&\qquad\qquad \left.-\frac{1}{2}\frac{d}{f\,(br)\,(br)^d}u^\mu u^\rho \partial_\rho u_\nu\right)
\end{aligned}
\tag{5.13}
$$

$$
K^{(0+1)} = K^{(0)}
\tag{5.14}
$$

We define the quantities with the superscript: $^{(0)}$, $^{(1)}$, $^{(2)}$ to be quantities that have no derivative, only one derivative or two derivative terms respectively. Now we insert (5.13) and (5.14) to Gauss-Codazzi equations (4.24) and get the constraint equations to the second order:

$$
\begin{aligned}
K_{\nu|\mu}^{\mu(0+1)} - K_{,\nu}^{(0+1)} &= K_{\nu|\mu}^{\mu(0)} - K_{,\nu}^{(0)} + K_{\nu|\mu}^{\mu(1)} - K_{,\nu}^{(1)} \\
&= \partial_\mu K_\nu^{\mu(0)} + \bar{\Gamma}_{\mu\sigma}^{\mu(1)} K_\nu^{\sigma(0)} - \bar{\Gamma}_{\mu\nu}^{\sigma(1)} K_\sigma^{\mu(0)} - \partial_\nu K^{(0)} + \\
&\quad +\partial_\mu K_\nu^{\mu(1)} + \bar{\Gamma}_{\mu\sigma}^{\mu(1)} K_\nu^{\sigma(1)} + \bar{\Gamma}_{\mu\sigma}^{\mu(2)} K_\nu^{\sigma(0)} \\
&\quad -\bar{\Gamma}_{\mu\nu}^{\sigma(1)} K_\sigma^{\mu(1)} - \bar{\Gamma}_{\mu\nu}^{\sigma(2)} K_\sigma^{\mu(0)} \\
&= \partial_\mu K_\nu^{\mu(0)} + \bar{\Gamma}_{\mu\sigma}^{\mu(1)} K_\nu^{\sigma(0)} - \bar{\Gamma}_{\mu\nu}^{\sigma(1)} K_\sigma^{\mu(0)} - \partial_\nu K^{(0)} + \\
&\quad +\frac{d}{2\,(bR)^d\,\sqrt{|f\,(bR)|}}\frac{2b}{d}\left(\partial_\mu\sigma_\nu^\mu - (d-1)\,A_\mu\sigma_\nu^\mu\right) = 0
\end{aligned}
\tag{5.15}
$$

where the bar over the Christoffel symbols represents the Christoffel symbols which have been calculated with respect to the metric $h_{\mu\nu}$ of the hyper-surface Φ. The fifth line is the LHS of equation (5.10).

Projecting (5.15) along u^ν we get,

$$\frac{d}{2\left(bR\right)^{d}\sqrt{|f\left(bR\right)|}}\left(\partial_{\mu}u^{\mu}-\left(d-1\right)u^{\nu}\partial_{\nu}\ln b-\frac{2b}{d}\sigma_{\alpha\beta}\sigma^{\alpha\beta}\right)=0\Rightarrow$$

$$\Rightarrow\partial_{\mu}u^{\mu}-\left(d-1\right)u^{\nu}\partial_{\nu}\ln b=\frac{2b}{d}\sigma_{\alpha\beta}\sigma^{\alpha\beta}\qquad(5.16)$$

where we used the relation,

$$
\begin{aligned}
-\partial_{\mu}\sigma_{\nu}^{\mu}u^{\nu} &= \sigma_{\nu}^{\mu}\partial_{\mu}u^{\nu}\\
&= \sigma_{\nu}^{\mu}P^{\mu\alpha}P_{\beta}^{\nu}\partial_{\alpha}u^{\beta}\\
&= \sigma_{\nu}^{\mu}P^{\mu\alpha}P_{\beta}^{\nu}\frac{1}{2}\left(\partial_{\alpha}u^{\beta}+\partial_{\beta}u^{\alpha}\right)\\
&= \sigma_{\nu}^{\mu}\left(P_{\mu}^{\alpha}P_{\beta}^{\nu}\frac{1}{2}\left(\partial_{\alpha}u^{\beta}+\partial_{\beta}u^{\alpha}\right)-P_{\mu}^{\nu}\frac{\partial_{\alpha}u^{\alpha}}{d-1}\right)\\
&= \sigma_{\alpha\beta}\sigma^{\alpha\beta}
\end{aligned}
$$

Projecting (5.15) along P_{σ}^{ν},

$$\frac{d}{2\left(bR\right)^{d}\sqrt{|f\left(bR\right)|}}$$

$$\times\left(-u^{\mu}\partial_{\mu}u_{\sigma}+P_{\sigma}^{\nu}\partial_{\nu}\ln b+\frac{2b}{d}P_{\sigma}^{\nu}\left(\partial_{\mu}\sigma_{\nu}^{\mu}-\left(d-1\right)A_{\mu}\sigma_{\nu}^{\mu}\right)\right)=0\Rightarrow$$

$$\Rightarrow u^{\mu}\partial_{\mu}u_{\sigma}-P_{\sigma}^{\nu}\partial_{\nu}\ln b=\frac{2b}{d}P_{\sigma}^{\nu}\left(\partial_{\mu}\sigma_{\nu}^{\mu}-\left(d-1\right)A_{\mu}\sigma_{\nu}^{\mu}\right)\qquad(5.17)$$

We can see that our results are in agreement with the equations of motion (2.5) of a conformal fluid to the second order in derivative expansion, that is, up to first order terms of the stress tensor (2.10).

5.2 Non-Constant Radial Location

We can generalize our discussion to non-constant hyper-surfaces, i.e.: $R=R\left(x\right)$. In order to do so, we need to use the property in which

we can explicitly calculate the r derivatives of tensors which have been calculated in the bulk. For instance, the constraint equations, $E_\mu^r = R_\mu^r = 0$ can have the following form:

$$R_\mu^r = \sum_i g_i\left(r, b\right) h_{i\mu}\left(\partial u\left(x\right), \partial b\left(x\right), \partial\partial u\left(x\right), \partial\partial b\left(x\right), higher\, terms\right)$$

(5.18)

Note that in this expression there are no derivatives of r because we can calculate them explicitly. However, derivatives of x^μ can not be calculated explicitly, because they are operating on arbitrary functions $u^\mu\left(x\right)$, $b\left(x\right)$.

For the second order constraint equations, we have no higher terms in the brackets of (5.18) and the answer for constant R was calculated above by Gauss-Codazzi equations (5.15). From (4.19) and the claim in chapter 4 (4.23) we have:

$$\left.\frac{1}{\mathcal{N}} R_\nu^{r(1+2)}\right|_{r=R} = K_{\nu|\mu}^{\mu(0+1)} - K_{,\nu}^{(0+1)} = g_1\left(R\right) h_{1\nu} = 0 \qquad (5.19)$$

where:

$$g_1\left(R\right) = \frac{d}{2\left(bR\right)^d \sqrt{\left|f\left(bR\right)\right|}} \qquad (5.20)$$

$$
\begin{aligned}
h_{1\nu} &= \Bigg(2du_\nu u^\mu \partial_\mu \ln b - u^\mu \partial_\mu u_\nu \\
&\quad -u_\nu \partial_\mu u^\mu + \left(d+1\right)\partial_\nu \ln b - dP_\nu^\mu \partial_\mu \ln b \\
&\quad +\frac{2b}{d}\left(\partial_\mu \sigma_\nu^\mu - \left(d-1\right) A_\mu \sigma_\nu^\mu\right)\Bigg)
\end{aligned}
\qquad (5.21)
$$

and

$$\mathcal{N} \;=\; \sqrt{\left| g^{ab} \partial_a \left(r - r\left(x^\alpha \right) \right) \partial_b \left(r - r\left(x^\alpha \right) \right) \right|} \qquad (5.22)$$

Therefore, the sum in (5.18) has only one term: $g_i\left(R\right) = 0 \; \forall i \geq 2$. However, the latter is supposed to be true for $\forall R = const$. Thus: $g_i\left(r\right) = 0 \; \forall i \geq 2$. Therefore, we obtain the constraint equations for the second order in a derivative expansion[1] which are the same for any non-null hyper-surface[2].

_____

[1] Because $g_1\left(R\right) \neq 0$, we can divide the equations: $g_1\left(R\right) h_{1\nu} = 0$ by $g_1\left(R\right)$.
[2] However, the constraint equations for $f\left(bR\right) = 0$, which is in the case of a null hyper-surface, hold true and are still the same as in the non-null case [6].
The case in which $R = 0$ is not define.

Chapter 6

Event vs Apparent Horizon

In this chapter we review the different notions of black hole horizons. We will focus our discussion on two distinct horizons, namely, the apparent and the event horizons. The last section in this chapter is devoted to calculating the constraint equations from the second order bulk metric projected on the different horizons.

6.1   Event Horizon

One approach to define the unique hyper-surface that defines the boundary of a black hole is the causal approach. In this approach one defines the unique hyper-surface, which is called the *event horizon,* as the boundary of the casual past of future null infinity [18]. This mathematical definition can be put in a more physical language by the following statement: the event horizon is the boundary of the region in space-time from which nothing can ever escape.

We should note here that the future null infinity is a time-like hyper-surface. That is because we deal with an asymptotically AdS space-time, for which the future directed null geodesics (which is the future null infinity in the definition) end on a time-like hyper-surface.

One should also note that by definition the event horizon is a null hyper-surface which is generated by null geodesics.

We would like to describe a method for finding the location of the event horizon. This method requires the knowledge of the late time generators of the event horizon, which means the location of the event horizon in late time. The method goes as follow: evolve the geodesic equation backwards in time and use the knowledge of the late time generators as the future boundary condition.

For our metric, which describes a dynamical black hole which is dual to viscous fluid flow, we can assume that after a while the fluid relaxes, which means that we have a stationary black hole that is described by the metric (3.2). The location of the event horizon for the metric is well known and is $r_{EH}^{stat} = \frac{1}{b}$. Since we are working in a small Knudsen number i.e., derivative expansion, we can find the location of the event horizon order by order [17]:

$$r_{EH}\left(x^{\alpha}\right) = \sum_{l=0}^{\infty} r_{EH}^{(l)}\left(x^{\alpha}\right) \tag{6.1}$$

We require that the normal vector ℓ^{a} to the event horizon hyper-surface $\mathcal{S}_{EH}$, defined by:

$$\mathcal{S}_{EH}\left(r, x\right) = r - r_{EH}\left(x\right), \ \ell^{a} = g^{ab}\partial_{b}\mathcal{S}_{EH} \tag{6.2}$$

will be a null vector $\ell^{a}\ell_{a} = 0$ and we get the equation,

$$g^{rr} - 2\partial_{\mu}r_{EH}g^{r\mu} + \partial_{\mu}r_{EH}\partial_{\nu}r_{EH}g^{\mu\nu} = 0 \tag{6.3}$$

which can be solved algebraically by using the initial condition for the location of the stationary event horizon: $r_{EH}^{(0)} = \frac{1}{b}$. The solution up to the second order,
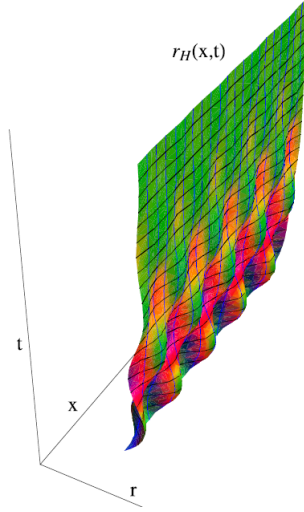
Figure 6.1: The event horizon $r = r_{EH}(x^\mu)$ is sketched as a function of the time t and one of the spatial coordinates x (the other $d-2$ spatial coordinates are suppressed). (Taken from [11]).

$$r_{EH}(x^\alpha) = \frac{1}{b(x)} + b(x)\left(h_1\sigma_{\alpha\beta}\sigma^{\alpha\beta} + h_2\omega_{\alpha\beta}\omega^{\alpha\beta} + h_3\mathcal{R}\right) \qquad (6.4)$$

with

$$
\begin{aligned}
h_1 &= \frac{2(d^2+d-4)}{d^2(d-1)(d-2)} - \frac{K_2(1)}{d(d-1)} \\
h_2 &= -\frac{d+2}{2d(d-2)} \\
h_3 &= -\frac{1}{d(d-1)(d-2)}
\end{aligned}
\qquad (6.5)
$$

Note that the solution is dependent on the boundary coordinates alone, which means that for a constant x^μ, i.e., a tube along the radial direction, we have a fixed solution and that in late time we get our starting condition that is the static solution (figure 6.1).

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6.2 Apparent Horizon

We would now like to describe a black hole by relating to its strong gravitational fields instead of its causal structure. The latter definition led us to the introduction of the event horizon hyper-surface in the previous section. In this section we show the major drawback in that definition, which leads us to a new hyper-surface, the apparent horizon [19].

In order to find the location of the event horizon, one needs to know the entire future evolution of space-time, which means that one needs to wait infinite time in order to find the location of the event horizon. To illustrate the problem, we will present the example given in [20]. Imagine a Schwarzschild black hole which has been irradiated by an infalling shell of null dust (figure 6.2). This is represented by the gray rectangle. The inward and outward null directions are tangent to the light gray dashed lines. Note that the inward null direction is horizontal. In order to find the event horizon, one needs to track the evolution of radial null geodesics. If they fall into the singularity, they are inside the black hole and if they are heading outward, even after the null dust passes, then they are outside the black hole. If one wants to provide the location of the event horizon without having any prior knowledge of the arrival of the null dust, one would state that it is located at the hard dashed line, which is clearly a false statement. One can realize this by examining the radial null geodesic B, which seems to be escaping from the black hole, but ends at the singularity. By a close examination one sees that the location of the event horizon is the hard line. One can see how it curves a bit in anticipation for the null dust to arrive, as if it "knows" that it is coming.

We will now explain how a new perspective on a dynamical black hole avoids the causal problem. This perspective is, as mentioned before, the strong gravitational field approach. In order to describe
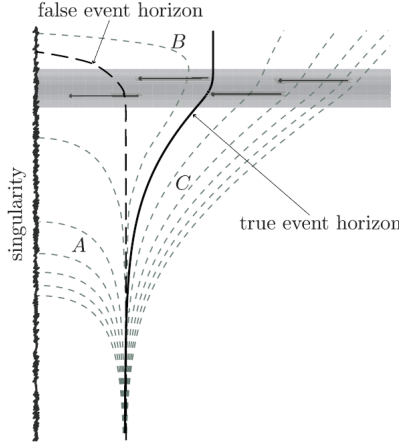
Figure 6.2: A schematic demonstrating the non-local nature of event horizon evolution for a spherically symmetric space-time with the angular dimensions suppressed. Horizontal location measures the radius of the associated spherical shell while time is (roughly) vertical. The shaded gray region represents infalling null dust. (Taken from [20]).

this approach we will give an example found in [19, 20]. Imagine a spherical shell, in a four dimensional world, which emits light and then stops. The light front null geodesics will be divided into two families that move in two null directions: the outgoing null direction and the ingoing null direction. We will denote the tangent to the light ray null vectors, which are perpendicular to the spherical shell surface, by ℓ^a and n^a respectively. The outgoing light front will increase in cross-sectional area and diverge, while the ingoing light front will decrease in the cross-sectional area and converge. In a more mathematical way, if we define the *expansion parameter* θ [14] as the fractional rate of change (per unit affine-parameter distance) of the congruence's (a family of geodesics that does not intersect each other) cross-sectional area:

$$\theta = \frac{1}{\delta A} \frac{d}{d\lambda} \delta A \qquad (6.6)$$
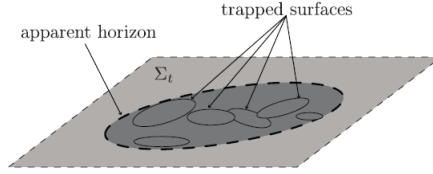
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Figure 6.3: An "instant" Σ_t along with some of its trapped surfaces (small black circles), the associated trapped region (dark gray) and the apparent horizon (thick dashed line). (Taken from [20]).

where λ is an affine-parameter, and δA is the infinitesimal cross-sectional area measured in the purely transverse directions[1], then we can describe the convergence and divergence of the light front by the following:

$$\theta_{(\ell)} > 0, \quad \theta_{(n)} < 0 \tag{6.7}$$

where the sub-script implies to which light front the expansion is referring to. A positive expansion means that the cross-sectional area will increase while negative expansion means that it will decrease.

Now we are ready to introduce the notion of a trapped surface: A *trapped surface* in $d + 1$ space-time dimensions is a space-like hyper-surface of $d - 1$ dimensions, which, by definition, has two null normals: the outgoing null normal ℓ^a and the ingoing null normal n^a, where the expansion in both directions is negative.

$$\theta_{(\ell)} < 0, \quad \theta_{(n)} < 0 \tag{6.8}$$

The hyper-surface that surrounds all of the trapped surfaces is called the *apparent horizon*[2] (figure 6.3) and defined mathematically by:

---

[1] The geodesics are crossing the cross-sectional area δA orthogonally.
[2] The apparent horizon is a special case of the so called *marginally trapped surface* reviewed in [21]

$$\theta_{(\ell)} = 0, \quad \theta_{(n)} < 0 \qquad (6.9)$$

Note to the condition $\theta_{(\ell)} = 0$. It does not mean that if matter comes into the black-hole the apparent horizon location will not change (increase)[3]. If we let the apparent horizon evolve in time, we get a co-dimension one tube (which means the dimensions of the tube is d). The tube is built in the following manner: given a foliation of space-time with $d-1$ hyper-surfaces S_λ which have two null normal vectors: ℓ^a and n^a as defined above, we connect them with an evolution vector field ν^a that maps every point in S_λ to $S_{\lambda+1}$ to create the d-dimensional tube Δ (figure 6.4). The evolution vector is perpendicular to each of the hyper-surface S_λ and tangent to the d-dimensional tube Δ. If the S_λ are the apparent horizon hyper-surfaces, then we say that Δ is a *time evolved apparent horizon*[4].

We will now explain about the geometric quantities that we defined. First we state that we cross normalized the null normal vectors,

$$\ell^a n_a = -1 \qquad (6.10)$$

Moreover, we present their normalization:

$$\ell^a \to f\ell^a \quad n^a \to \frac{1}{f}n^a \qquad (6.11)$$

where f is an arbitrary function.

Then we define the evolution vector to be normal to the S_λ hyper-surfaces, so it will be built from the two null vectors[5] ℓ^a and n^a:

---

[3]The expansion of the apparent horizon is related the expansion in the direction of the evolution vector ν^a (6.12), which is:

$$\theta_{(\nu)} = \theta_{(\ell)} - C\theta_{(n)} = -C\theta_{(n)} > 0$$

[4]From now on we will refer to the time evolved apparent horizon as the apparent horizon.
[5]Because we are using null vectors, we can always choose a normalization that will give us this form of evolution vector.
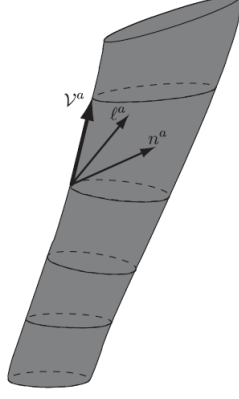
Figure 6.4: A schematic of an d-tube Δ with compact foliation hyper-surfaces S_λ along with the outward and inward pointing null normals to those hyper-surfaces. ν^a is the future-pointing tangent to Δ that is simultaneously normal to the S_λ. (Taken from [20]).

$$\nu^a = \ell^a - Cn^a \tag{6.12}$$

for some function C. Note that if $C > 0$, then we get space-like hyper-surface, if $C = 0$, we get a null hyper-surface and if $C < 0$, we get a time-like hyper-surface.

Now we can present m^a, the normal to the d-tube hyper-surface Δ vector, which is perpendicular to the evolution vector and we get:

$$m^a = \ell^a + Cn^a \tag{6.13}$$

Because m^a is normal to the hyper-surface Δ, which can be presented by the hyper-surface equation $\phi(r, x) = 0$, we will choose to define it as:

$$m^a = g^{ab}\partial_b\phi \tag{6.14}$$

which is similar to what we did in (4.1) but without having to normalize it[6].

---

[6] Note that m^a can be space-like, time-like and even null.

We pause here and return to the event horizon description. The event horizon can also be described by the d-tube description and of course its evolution vector will be with $C = 0$, so we actually see that both the normal vector to the hyper-surface Δ (the event horizon) and the evolution vector are the same:

$$m^a = \nu^a = \ell^a \tag{6.15}$$

where ℓ^a is defined in (4.3)[7]

We now specify some more geometric quantities and we introduce the metric for the hyper-surface S_λ, as well:

$$q_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a \tag{6.16}$$

Note that this metric has two zero eigenvalues.

We can associate the expansion parameters to the geometric quantities using the following definition:

$$\theta_{(\ell)} = q^{ab}\nabla_a \ell_b, \quad \theta_{(n)} = q^{ab}\nabla_a n_b \tag{6.17}$$

Note that q^{ab} is defined as in (6.16), but with upper indices and it is not the inverse of q_{ab}[8].

Now we are ready to return to the causal problem (the global definition of the event horizon) presented earlier and see whether we can provide a solution to it. One can see (figure 6.5) that the apparent horizon changes only when the null dust comes in, as opposed to the event horizon that "anticipates" the arrival of the null dust, and curves even before it arrives. Therefore, we have found a solution to the problem. Note that the apparent horizon is located inside the black hole, which means that it lies within the event horizon. Moreover, the apparent and the event horizons are the same

---

[7]In the rest of the section we will set the normalization of f in (4.3) to one.

[8]One can think of q^{ab} as a projection onto the S_λ hyper-surfaces, which in (6.17) restricts $\nabla_a \ell_b$ and $\nabla_a n_b$ to the hyper-surfaces. The expansion θ is the trace after the restriction.
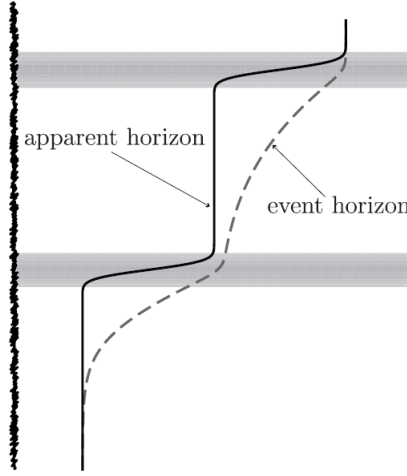
Figure 6.5: A simulation similar to that of Figure 4 though this time two distinct shells fall into the black hole. Both the apparent and event horizons are plotted. (Taken from [20]).

in the stationary case; one can see this by looking at their location after the two null dusts pass (figure 6.5).

We want to find the location of the apparent horizon in the same manner that we did for the event horizon, by using a derivative expansion:

$$r_{AH}\left(x^{\alpha}\right) = \sum_{l=0}^{\infty} r_{AH}^{(l)}\left(x^{\alpha}\right) \tag{6.18}$$

From the requirement $\theta_{(\ell)} = 0$ we can find the location order by order.

However, in order to do so we need to foliate space-time with $d-1$ hyper-surfaces S_{λ}. We define those surfaces by specifying the two null vectors to the surfaces. The symmetry of space-time singles out the S_{λ} hyper-surfaces of constant r and constant "time" propagation. The latter is an arbitrary combination of x^{μ}. The outward null normal ℓ^{a} and the inward null normal n^{a} are chosen to

be:

$$\ell^a = \mathcal{B} \left(\frac{\partial}{\partial r} \right)^a + \mathcal{N}^\mu \left(\frac{\partial}{\partial x^\mu} \right)^a \tag{6.19}$$

$$n^a = - \left(\frac{\partial}{\partial r} \right)^a \tag{6.20}$$

We can determined them by the relation (6.10) and the fact that they are both null vectors and we get

$$\mathcal{N}^\mu = u^\mu + \mathcal{T}^\mu, \quad \mathcal{T}^\mu u_\mu = 0 \tag{6.21}$$

$$\mathcal{B} = -\mathcal{N}^\mu \mathcal{V}_\mu \tag{6.22}$$

to the second order, where $\mathcal{T}^\mu$ is composed from one or two derivatives of the d-velocity or the temperature, so we have freedom in determining this quantity. We therefore require that this description be suited to describe the event horizon, because we know that the apparent horizon coincides with the event horizon in the stationary case. In order to do so, we require that the null outward vector in the x^μ direction resembles the null outward vector in the x^μ direction that defines the event horizon. This requirement determines $\mathcal{T}^\mu$, and we get:

$$\begin{aligned}
\mathcal{T}^\mu &= \left(1 - \frac{1}{br} \right) b u^\lambda \partial_\lambda u^\mu - \frac{1}{r^2} \frac{1}{d-2} P^{\mu\rho} \left(\partial_\lambda - (d-3) A_\lambda \right) \omega_\rho^\lambda + \\
&\quad + \left(\frac{1}{r^2} \left(2L\,(br) + \frac{1}{d-2} \right) - \frac{2b^2}{d} \right) P^{\mu\rho} \left(\partial_\lambda - (d-1) A_\lambda \right) \sigma_\rho^\lambda + \\
&\quad + \left(1 - \frac{1}{br} \right) \frac{2b}{r} F\,(br) \, \sigma^{\mu\nu} u^\lambda \partial_\lambda u_\nu
\end{aligned} \tag{6.23}$$

Now we are ready to find the expansion in the outward null di-

rection:

$$
\begin{aligned}
\theta_{(\ell)} &= q^{ab}\nabla_a \ell_b \\
&= \nabla_a \ell^a + \ell^a n^b \nabla_a \ell_b \\
&= \partial_a \ell^a + \Gamma^a_{ab}\ell^b + \ell^a n^b \left(\partial_a \ell_b - \Gamma^c_{ab}\ell_c\right) \\
&= \partial_r \ell^r + \partial_\mu \ell^\mu + 2\Gamma^r_{r\mu}\ell^\mu + \Gamma^\nu_{\nu r}\ell^r + \Gamma^\nu_{\nu\mu}\ell^\mu + \ell^\nu \Gamma^\mu_{\nu r}\ell_\mu \\
&= -\frac{(d-1)}{2r\,(br)^d}\left[r^2 \left(1 - (br)^d\right) - \left(\frac{2\,(br)^d}{(d-2)} + \frac{1}{2}\right)\omega_{\alpha\beta}\omega^{\alpha\beta} + \right. \\
&\quad \left. + \left(\frac{2\,(br)^{d-2}}{d-2} - \frac{K_2\,(br)}{d-1}\right)(br)^2\,\sigma_{\alpha\beta}\sigma^{\alpha\beta}\right] + \frac{\mathcal{R}}{2r\,(d-2)} \\
&\quad -\frac{r^2 \left(1 - (br)^d\right)}{2\,(br)^d}\left(-b^2 \partial_r K_1\,(br)\,\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \frac{1}{r^3}\omega_{\alpha\beta}\omega^{\alpha\beta}\right) (6.24)
\end{aligned}
$$

Inserting (6.18) to (6.24) and using $\theta_{(\ell)} = 0$, we find order by order the location of the apparent horizon:

$$
r_{AH}\,(x^\alpha) = \frac{1}{b\,(x)} + b\,(x)\left(\widetilde{h}_1 \sigma_{\alpha\beta}\sigma^{\alpha\beta} + h_2 \omega_{\alpha\beta}\omega^{\alpha\beta} + h_3 \mathcal{R}\right) \quad (6.25)
$$

$$
\widetilde{h}_1 = h_1 - \frac{4}{d^2\,(d-1)} \quad (6.26)
$$

where h_1, h_2, h_3 are defined in (6.5).

Note that up to the first order in derivative expansion, the event and apparent horizons coincide.

6.3 Event and Apparent Horizons Constraint Equations

We are now well equipped to check if the constraint equations for the event and apparent horizons are the same up to the third order in a derivative expansion. The work that was done in [6] showed that

the event horizon, to the first order in derivative expansion of the metric, exhibits the same constraint equations to the second order in derivative expansion[9], as in the case of non-null hyper-surface (5.16), (5.17). As we mentioned above, the event and apparent horizons differ only in the second order in derivative expansion. Therefore, the constraint equations up to the second order are the same in all of the hyper-surfaces mentioned above. Recall our statement about the ability to get the same constraint equations from different hyper-surfaces[10], we see that it turns out to be right up to first order in the metric.

In order to check this statement for the event and apparent horizons to the second order in derivative expansion of the metric, let us look directly at the constraint equations i.e.: E_μ^r and calculate their difference while they are evaluated on the event and apparent horizons hyper-surfaces:

$$EH - AH = E_\mu^r\big|_{r=r_{EH}} - E_\mu^r\big|_{r=r_{AH}} = R_\mu^r\big|_{r=r_{EH}} - R_\mu^r\big|_{r=r_{AH}} \quad (6.27)$$

Bear in mind that the difference in R_μ^r between the event and apparent horizons to the third order will only come from the evaluation of R_μ^r in the bulk to the first order and from the location of the horizons to the second order. Recall that the difference between the horizons location is only in the second order.

Now we present the calculation of the tensor R_μ^r to the first order in the bulk:

$$R_\mu^{r(1)} = \frac{1}{2} r \frac{d}{(br)^d} \left(\partial_\mu \ln b + d u^\rho \partial_\rho \ln b - u_\mu \partial_\alpha u^\alpha - u^\nu \partial_\nu u_\mu \right) \quad (6.28)$$

---

[9]The constraint equations always have one derivative more than the derivative expansion of the metric

[10]Actually for hyper-surfaces of the form $\phi = r - r(x^\alpha)$.

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From Einstein's equations (3.1) we know that $R_\mu^{r(1)} = 0$ for any r. Therefore, the expression in the brackets in (6.28) is zero. This expression is just the constraint equations we found in (5.11) and (5.12).

Now we are ready to calculate (6.27) to the third order:

$$
\begin{aligned}
EH - AH &= R_\mu^{r(1)}\big|_{r=r_{EH}} - R_\mu^{r(1)}\big|_{r=r_{AH}} \\
&= \left(\frac{1}{2} r_{EH} \frac{d}{(br_{EH})^d} - \frac{1}{2} r_{AH} \frac{d}{(br_{AH})^d} \right) (\partial_\mu \ln b \\
&\quad + du^\rho \partial_\rho \ln b - u_\mu \partial_\alpha u^\alpha - u^\nu \partial_\nu u_\mu) \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.29)
\end{aligned}
$$

where we get the fourth line by using the constraint equation to the first order.

There is another method to get this result. This method is more cumbersome than what we presented here, however, it poses challenging computational techniques; therefore, we presented it in the appendix.

Chapter 7

Hydrodynamic Quantities

7.1 Entropy Current

In [17] the authors introduce a boundary entropy current obtained from the event horizon by using the area of the black hole event horizon. By using the connection between the area of a hyper-surface to the square root of the metric determinant of that hyper-surface, and by using a pull-back along ingoing null geodesics to that area, we get the entropy current on the boundary[1]. In order to do so, we split the boundary coordinate x^μ to (ν, x^i) and we continue to use the same coordinates on the hyper-surface $\Phi = r - r(x)$. We will get, as in the d-tube description, a space-like hyper-surfaces Σ_ν which is a constant ν slice. This, as well, will divide the components of m^μ into (m^ν, m^i)[2]. After we split space-time, we define the metric on the space-like hyper-surface Σ_ν by the restriction of the line element to be only on that hyper-surface:

$$ds^2 = g_{ab}dx^a dx^b = \widetilde{q}_{ij}dx^j dx^j \tag{7.1}$$

---

[1] Of course not every hyper-surface is adequate. We require that the hyper-surface admits the area increase theorem, that gives us a natural candidate: the event horizon. However, it is not the only hyper-surface with that property.

[2] Here ν does not represent d different coordinates, but a specific coordinate. These notations were also used in [17].

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Then by using the procedure done in [17]we get:

$$J^\mu = \frac{\sqrt{\widetilde{q}}}{4} \frac{m^\mu}{m^\nu} \tag{7.2}$$

where $\widetilde{q}$ is the determinant of $\widetilde{q}_{ij}$. It can be shown that this form of entropy current is indeed covariant [17]. Also the authors in [17] found the entropy current for the event horizon hyper-surface, and [13] reproduced it to arbitrary d-dimensions:

$$
\begin{aligned}
4b^{d-1} J^\mu_{EH} &= u^\mu + b^2 u^\mu \left[A_1 \sigma_{\alpha\beta}\sigma^{\alpha\beta} + A_2 \omega_{\alpha\beta}\omega^{\alpha\beta} + A_3 \mathcal{R} \right] + \\
&+ b^2 \left[B_1 D_\lambda \sigma^{\mu\lambda} + B_2 D_\lambda \omega^{\mu\lambda} \right]
\end{aligned}
\tag{7.3}
$$

with,

$$
\begin{aligned}
A_1 &= \frac{2}{d^2}(d+2) - \frac{K_1(1)\,d + K_2(1)}{d}, \quad A_2 = -\frac{1}{2d}, \quad B_2 = \frac{1}{d-2}, \\
B_1 &= -2A_3 = \frac{2}{d(d-2)}
\end{aligned}
\tag{7.4}
$$

In order to evaluate the the entropy current to the second order in derivative expansion for the apparent horizon, we calculate the difference between the two entropy currents:

$$J^\mu_{EH} - J^\mu_{AH} = \frac{\sqrt{\widetilde{q}}}{4}\frac{m^\mu}{m^\nu}\Bigg|_{r=r_{EH}} - \frac{\sqrt{\widetilde{q}}}{4}\frac{m^\mu}{m^\nu}\Bigg|_{r=r_{AH}} \tag{7.5}$$

Because m^μ for the event horizon and apparent horizon is the same (C.5), we need to evaluate the difference in $\sqrt{\widetilde{q}}$ on the horizons. The difference to the order we are seeking will come from inputting the location of the horizon to the second order into the zero order of the metric determinant,

$$\sqrt{\widetilde{q}}^{(0)} = r^{d-1} \tag{7.6}$$

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and we get the difference between the entropy currents to be:

$$
\begin{aligned}
J^\mu_{EH} - J^\mu_{AH} &= \frac{1}{4}\frac{m^\mu}{m^\nu}\left(\sqrt{\tilde{q}}\Big|_{r=r_{EH}} - \sqrt{\tilde{q}}\Big|_{r=r_{AH}}\right) \\
&= \frac{1}{4}\frac{m^\mu}{m^\nu}\left(r^{d-1}\Big|_{r=r_{EH}} - r^{d-1}\Big|_{r=r_{AH}}\right) \\
&= \frac{(d-1)}{4}\frac{m^\mu}{m^\nu}b^{-(d-2)}\left(r^{(2)}\Big|_{r=r_{EH}} - r^{(2)}\Big|_{r=r_{AH}}\right) \\
&= \frac{m^\mu}{m^\nu}\frac{b^{-(d-3)}}{d^2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} \quad\quad\quad (7.7)
\end{aligned}
$$

Therefore, we see that the entropy current for the apparent horizon is,

$$
\begin{aligned}
4b^{d-1}J^\mu_{AH} &= u^\mu + b^2 u^\mu\left[\widetilde{A}_1\sigma_{\alpha\beta}\sigma^{\alpha\beta} + A_2\omega_{\alpha\beta}\omega^{\alpha\beta} + A_3\mathcal{R}\right] + \\
&+ b^2\left[B_1 D_\lambda\sigma^{\mu\lambda} + B_2 D_\lambda\omega^{\mu\lambda}\right] \quad\quad\quad (7.8)
\end{aligned}
$$

with,

$$
\widetilde{A}_1 = A_1 - \frac{4}{d^2} \quad\quad\quad (7.9)
$$

and the other coefficients are the same as the coefficients of the event horizon entropy current (7.4).

We now have to check if this entropy current be suitable to describe the notion of entropy, which means we need to answer the question: does the entropy current never decreases?

In order to answer this question [17] calculated the divergence of the entropy current and required the non-negativeness of the expression:

$$
D_\mu J^\mu \geq 0 \quad\quad\quad (7.10)
$$

They did it with arbitrary coefficients A_1, A_2, A_3, B_1, B_2 and found that in order that (7.10) applies, the following connection should be valid:

$$B_1 = 2A_3 \tag{7.11}$$

This connection, of course, is holds true for the event horizon and also for the apparent horizon. Therefore, we can say that it is possible to ascribe a different entropy current to the boundary by the apparent horizon hyper-surface.

7.2   Temperature

Besides the entropy current, we can relate another thermodynamic quantity of the fluid to gravity, that is the temperature.

The temperature of the black hole can be related to the surface gravity of the event horizon by $T_{EH} = \frac{\kappa}{2\pi}$. We use a generalized definition for the surface gravity shown in [20]:

$$\kappa_\nu = -n_b \nu^a \nabla_a \ell^b = -n_b \ell^a \nabla_a \ell^b + C n_b n^a \nabla_a \ell^b = \kappa_\ell + C\kappa_n \tag{7.12}$$

to calculate the surface gravity for different hyper-surfaces.

The result for a hyper-surface located at $r = \frac{1}{b} + r^{(2)}$ ($r^{(2)}$ has second order derivatives) is:

$$
\begin{aligned}
\kappa_\nu &= \kappa_\ell \\
&= \frac{d}{2b} - \frac{\partial_\lambda u^\lambda}{d-1} \\
&\quad - \frac{d}{2}(d-3)\, r^{(2)} + \partial_r \left[\frac{1}{2(br)^d} \left(\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} \right. \right. \\
&\qquad \left. \left. + (br)^2 K_2(br) \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} \right) \right]
\end{aligned}
\tag{7.13}
$$

and by inserting the location of the event horizon or the apparent horizon, we can get their associated surface gravity respectively. These quantities may relate to different notion of temperature fields by dividing them by 2π.

In order to set the surface gravity to be equal, up to 2π, to the temperature field of the conformal fluid we set the normalization of (6.11) in such a way that we get: $T_{fluid} = \frac{\kappa_\ell}{2\pi}$. The surface gravity transforms in the following way:

$$
\kappa'_\ell = f\left(\kappa_\ell + \frac{d}{d\lambda} \ln f \right)
\tag{7.14}
$$

where f is the normalization in (6.11), κ'_ℓ is the new surface gravity and λ is an arbitrary parameter[3].

Under this change of the normalization f, we need to check how the other quantities that we defined transform and if this transformation changes our results regarding the location of the apparent horizon and the entropy current.

The location of the apparent horizon depends on how the expansion parameter transforms:

$$
\theta'_{(\ell)} = f\, \theta_{(\ell)}
\tag{7.15}
$$

---

[3]Note that this is the transformation of the change of the parameter λ, because if $\lambda \to \lambda^*$, then $\ell^{*a} = \frac{d\lambda}{d\lambda^*}\ell^a \equiv f\,\ell^a$.

and we can see that it does not change the location of the apparent horizon. We can see this by trying to find the "new" location of the apparent horizon by setting $\theta'_{(\ell)} = 0$. It means that if $f \neq 0$, which is the trivial case, then $\theta_{(\ell)} = 0$ which is exactly the "old" location of the apparent horizon.

By the definition of the entropy current (7.2) one can see that it is invariant to the normalization transformation.

Chapter 8

Conclusions

In this thesis we presented a method and different techniques to evaluate properties of an out-of-equilibrium black hole that has a dual description as a conformal out-of-equilibrium relativistic fluid, which "lives" on the boundary of AdS space-time. We set the stage and presented the derivative expansion for both the relativistic fluid and for the black-hole. We presented a way to compute the Einstein constraint equations for different hyper-surfaces. The constraint equations turned out to be just the conservation equations of the fluid. We found, as expected, that from any hyper-surface which is located at $r = r(x)$, we get the same conservation equations. This implies that maybe one can describe a hydrodynamic duality, not just from the quantities that are related to a certain hyper-surface, namely the event horizon, but also to quantities related to other hyper-surfaces. We suggested the apparent horizon as a possible candidate, and calculated from the apparent horizon area the entropy current and from its surface gravity the temperature for a conformal fluid. Our work poses a few future questions. The first is: which hyper-surface describes correctly the conformal fluid properties? The second is: how can we use these results for the description of the QGP (Quark Gluon Plasma)?

Our solution for the difference between the location and the entropy current of the event and apparent horizons is in agreement with the boost invariant Bjorken flow which was done in [20, 23]. Only recently we came across a similar work [22] that finds the entropy for the apparent horizon. They used a slightly different approach than what we did to define the entropy current for the apparent horizon. However, it is in complete agreement with our results.

Moreover, a recent work [23] suggests that the apparent horizon is a better candidate to describe the dual fluid flow at the boundary. This work examines the exact solution of the conformal soliton flow, which results in the divergence of the area of the event horizon, while the apparent horizon stays finite and constant. This suggests that the corresponding entropy of the fluid should be described by the apparent horizon.

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Appendix A

Notation

We refer to lower Roman indices that start from the beginning of the alphabet $a, b \ldots, h$ as the whole $d + 1$ space-time coordinates (bulk coordinate). The Greek indices $\mu, \nu...$ refer to the d dimensions hyper-surface of the fluid that "lives" on the boundary of the bulk, which are just the indices of the CFT. The lower Roman indices that start from $i, j...$ refer to the $d - 1$ spatial directions of the fluid.

We commonly use different metrics, which we will specify here:

- g_{ab} - is the $d + 1$-dimensional bulk metric. We use this metric to lower and raise tensors with bulk indices that are defined in the whole space-time.

- h_{ab} - is a $d + 1$-dimensional metric that is restricted to a certain hyper-surface. We use this metric to lower and raise tensors with bulk indices that are defined on a certain hyper-surface.

- $h_{\mu\nu}$ - is a d-dimensional metric of a hyper-surface. We get this metric by a projection of g_{ab} into the hyper-surface. We use this metric to lower and raise tensors with Greek indices that are defined on a certain hyper-surface.

- $g_{\mu\nu}$ - is a d-dimensional boundary metric. In our calculation $g_{\mu\nu}$ is the flat metric $\eta_{\mu\nu}$. Only in the section that deals with

the Weyl formalism, do we let it be also a curved metric.

- $\eta_{\mu\nu}$ - is the d-dimensional Minkowski boundary metric, with the signature convention of "more plus signs" $(-,+,+,...)$. This is the metric of the fluid in the boundary and we will lower and raise tensors with boundary Greek indices with that metric, i.e., $u^{\mu} = \eta^{\mu\nu}u_{\nu}$.

- q_{ab} - is a $d+1$-dimensional metric, which defines the $d-1$ space-like hyper-surfaces that create the d-dimensional tube. This metric projects the bulk tensor onto the $d-1$ hyper-surface. We use this metric to lower and raise tensors that we want to restrict into the $d-1$ hyper-surface and that are written in the bulk indices, $a, b, ...$

- $\widetilde{q}_{ij}$ - is a $d-1$-dimensional metric, which defines the $d-1$ space-like hyper-surfaces that create the d-dimensional tube. We use this metric to lower and raise tensors that are restricted to the $d-1$ hyper-surface and have $i, j...$ indices.

We remind the reader of the basic tensors used commonly in this thesis:

The *Riemann curvature tensor* defined below for bulk vector field A^a,

$$\nabla_c \nabla_d A^a - \nabla_d \nabla_c A^a = R^a_{bcd} \quad A^b \tag{A.1}$$

$$R^a_{bcd} \equiv \Gamma^a_{bd,c} - \Gamma^a_{cb,d} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} \tag{A.2}$$

Additionally we define the *Ricci tensor* and the *Ricci scalar* by,

$$R_{bd} = R^a_{bad} \tag{A.3}$$

$$R = g^{ab}R_{ab} \tag{A.4}$$

Appendix B

Detailed Calculations

For the hyper-surface $\Phi = r - R$, where R is constant, we have the basis vectors written in (5.3). Then the metric on the surface, to any order, is:

$$h_{\mu\nu} = g_{ab}e^a_\mu e^b_\nu = g_{\mu\nu} \tag{B.1}$$

The inverse metrics to the zero and the first order are:

$$h^{\mu\nu}_{(0)} = \frac{1}{R^2}P^{\mu\nu} - \frac{1}{R^2 f(bR)}u^\mu u^\nu \tag{B.2}$$

$$
\begin{aligned}
h^{\mu\nu}_{(0+1)} &= h^{\mu\nu}_{(0)} - \frac{2bF(bR)}{R^2}\sigma^{\mu\nu} - \frac{2}{d-1}\frac{1}{R^3 f(bR)^2}\partial_\lambda u^\lambda u^\mu u^\nu \\
&+ \frac{1}{R^3 f(bR)}u^\lambda \partial_\lambda(u^\mu u^\nu) \tag{B.3}
\end{aligned}
$$

The Christoffel symbol, with respect to the metric of the hyper-surface Φ to the first order, is:

$$\bar{\Gamma}^{\sigma(1)}_{\mu\nu} = \tfrac{1}{2}h^{\sigma\lambda} \quad (h_{\mu\lambda,\nu} + h_{\nu\lambda,\mu} - h_{\mu\nu,\lambda})$$

$$= \frac{1}{2\,(bR)^d}\left[\frac{1}{2}u_{(\mu}\partial_{\nu)}u^\sigma + du_\mu u_\nu P^{\sigma\lambda}\partial_\lambda \ln b\right.$$

$$-P^{\sigma\lambda}\partial_\lambda(u_\mu u_\nu) + \frac{1}{f\,(bR)}u^\sigma\left(\frac{1}{2}\partial_{(\nu}u_{\mu)}\right)$$

$$-\frac{1}{2}du_{(\mu}\partial_{\nu)}\ln b - du^\lambda u_\mu u_\nu \partial_\lambda \ln b$$

$$\left.+u^\lambda \partial_\lambda(u_\mu u_\nu)\Big)\right] \tag{B.4}$$

$$\bar{\Gamma}^{\mu(2)}_{\mu\nu} = \frac{1}{d-1}\frac{1}{Rf\,(bR)}\left(\partial_\rho u^\rho \partial_\nu \ln f\,(bR) - \partial_\nu \partial_\rho u^\rho\right) \tag{B.5}$$

$$\bar{\Gamma}^{\sigma(2)}_{\mu\nu}u^\mu u_\sigma = \frac{1}{2}\frac{1}{Rf\,(bR)}\left(u_\nu u^\rho \partial_\rho u^\lambda \partial_\lambda f\,(bR) + 2\frac{\partial_\nu \partial_\rho u^\rho}{d-1}\right.$$

$$-2u^\rho \partial_\rho u^\lambda \partial_\nu u_\lambda - 2\frac{\partial_\rho u^\rho}{d-1}\partial_\nu \ln f\,(bR)$$

$$+ (1 - f\,(bR))\,u^\rho\left(\partial_\rho u^\lambda \partial_\nu u_\lambda\right.$$

$$\left.\left.-u_\nu \partial_\rho u^\lambda u^\mu \partial_\mu u_\lambda - \partial_\rho u^\lambda \partial_\lambda u_\nu\right)\right) \tag{B.6}$$

We will specify the detailed calculation of the Christoffel symbols for the entire bulk space-time.

The Christoffel symbols that we can calculate in full to all orders, of the full bulk metric g_{ab}, are:

$$\Gamma^\nu_{rr} = \frac{1}{2}g^{\nu a}\left(g_{ra,r} + g_{ra,r} - g_{rr,a}\right)$$

$$= \frac{1}{2}g^{\nu r}\left(g_{rr,r} + g_{rr,r} - g_{rr,r}\right)$$

$$+\frac{1}{2}g^{\nu\lambda}\left(g_{r\lambda,r} + g_{r\lambda,r} - g_{rr,\lambda}\right) = 0 \tag{B.7}$$

$$
\begin{aligned}
\Gamma^r_{rr} &= \frac{1}{2}g^{ra}\left(g_{ra,r} + g_{ra,r} - g_{rr,a}\right) \\
&= \frac{1}{2}g^{rr}\left(g_{rr,r} + g_{rr,r} - g_{rr,r}\right) \\
&\quad + \frac{1}{2}g^{r\lambda}\left(g_{r\lambda,r} + g_{r\lambda,r} - g_{rr,\lambda}\right) = 0 \qquad \text{(B.8)}
\end{aligned}
$$

The Christoffel symbols that are needed to calculate the Ricci tensor, with respect to the metric $g_{ab}^{(0+1)} = g_{ab}^{(0)} + g_{ab}^{(1)}$, to the first order are:

$$
\begin{aligned}
\Gamma^{\nu(0+1)}_{\mu\sigma} &= \frac{1}{2}g^{\nu a}\left(g_{\mu a,\sigma} + g_{\sigma a,\mu} - g_{\mu\sigma,a}\right) \\
&= \frac{1}{2}g^{\nu r}\left(g_{\mu r,\sigma} + g_{\sigma r,\mu} - g_{\mu\sigma,r}\right) \\
&\quad + \frac{1}{2}g^{\nu\lambda}\left(g_{\mu\lambda,\sigma} + g_{\sigma\lambda,\mu} - g_{\mu\sigma,\lambda}\right) \\
&= \frac{1}{2}u^{\nu}\left(-2\partial_{(\mu}u_{\sigma)} - \partial_r\left(-r^2 f\left(br\right)u_\mu u_\sigma + r^2 P_{\mu\sigma}\right)\right) \\
&\quad + \frac{1}{2}\left[2\left(1 - f\left(br\right)\right)u_{(\mu}\partial_{\sigma)}u^{\nu}\right. \\
&\qquad \left. - P^{\nu\rho}\partial_\rho\left(-f\left(br\right)u_\mu u_\sigma + P_{\mu\sigma}\right)\right] \\
&\quad - b\partial_r\left(r^2 F\left(br\right)\right)u^{\nu}\sigma_{\mu\sigma} - \frac{\partial_\lambda u^\lambda}{d-1}u^{\nu}u_\mu u_\sigma \\
&\quad + \frac{1}{2}u^{\nu}u^{\rho}\partial_\rho\left(u_\mu u_\sigma\right) - \frac{1}{2r}\partial_r\left(r^2 f\left(br\right)\right)u_\mu u_\sigma u^{\rho}\partial_\rho u^{\nu} \\
&\quad + u^{\rho}\partial_\rho u^{\nu}P_{\mu\sigma} \qquad \text{(B.9)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\nu(0+1)}_{\mu r} &= \frac{1}{2}g^{\nu a}\left(g_{\mu a,r} + g_{ra,\mu} - g_{\mu r,a}\right) \\
&= \frac{1}{2}g^{\nu r}\left(g_{\mu r,r} + g_{rr,\mu} - g_{\mu r,r}\right) + \frac{1}{2}g^{\nu\lambda}\left(g_{\mu\lambda,r} + g_{r\lambda,\mu} - g_{\mu r,\lambda}\right) \\
&= \frac{1}{r}P^{\nu}_\mu + \frac{1}{2r^2} + \omega^{\nu}_\mu + b\partial_r\left(F\left(br\right)\right)\sigma^{\nu}_\mu \qquad \text{(B.10)}
\end{aligned}
$$

$$\Gamma^{r(0+1)}_{\mu\nu} = \frac{1}{2}g^{ra}\left(g_{\mu a,\nu} + g_{\nu a,\mu} - g_{\mu\nu,a}\right)$$

$$= \frac{1}{2}g^{rr}\left(g_{\mu r,\nu} + g_{\nu r,\mu} - g_{\mu\nu,r}\right) + \frac{1}{2}g^{r\lambda}\left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right)$$

$$= \frac{1}{2}r^2 f\left(br\right)\left(-2\partial_{(\mu}u_{\nu)} - \partial_r\left(-r^2 f\left(br\right)u_\mu u_\nu + r^2 P_{\mu\nu}\right)\right)$$

$$+ \frac{1}{2}r^2\left[2\partial_{(\nu}\left(f\left(br\right)u_{\mu)}\right)\right.$$

$$-2\partial_{(\nu}u_{\mu)} - u^\rho\partial_\rho\left(-f\left(br\right)u_\mu u_\nu + P_{\mu\nu}\right)\right]$$

$$+ r^2 f\left(br\right)\left(-b\partial_r\left(r^2 F\left(br\right)\right)\sigma_{\mu\nu}\right.$$

$$-\frac{\partial_\lambda u^\lambda}{d-1}u_\mu u_\nu + \frac{1}{2}u^\lambda u_\lambda\left(u_\mu u_\nu\right)\right)$$

$$-\frac{\partial_\lambda u^\lambda}{d-1}r\partial_r\left(r^2 f\left(br\right)\right)u_\mu u_\nu + 2r^2\frac{\partial_\lambda u^\lambda}{d-1}P_{\mu\nu} \qquad (B.11)$$

$$\Gamma^{r(0+1)}_{\mu r} = \frac{1}{2}g^{ra}\left(g_{\mu a,r} + g_{ra,\mu} - g_{\mu r,a}\right)$$

$$= \frac{1}{2}g^{rr}\left(g_{\mu r,r} + g_{rr,\mu} - g_{\mu r,r}\right)$$

$$+ \frac{1}{2}g^{r\lambda}\Gamma^{r(0+1)}_{\mu\nu}\left(g_{\mu\lambda,r} + g_{r\lambda,\mu} - g_{\mu r,\lambda}\right)$$

$$= \frac{1}{2}\partial_r\left(r^2 f\left(br\right)\right)u_\mu - \frac{\partial_\lambda u^\lambda}{d-1}u_\mu \qquad (B.12)$$

The Christoffel symbols that are needed to calculate the expansion parameter $\theta_{(\ell)}$, with respect to the metric $g_{ab}^{(0+1+2)} = g_{ab}^{(0)} + g_{ab}^{(1)} + g_{ab}^{(2)}$

to the second order, are:

$$
\begin{aligned}
\Gamma^{r(0+1)}_{\mu\nu} &= \frac{1}{2}g^{ra}\left(g_{\mu a,\nu} + g_{\nu a,\mu} - g_{\mu\nu,a}\right) \\
&= \frac{1}{2}g^{rr}\left(g_{\mu r,\nu} + g_{\nu r,\mu} - g_{\mu\nu,r}\right) + \frac{1}{2}g^{r\lambda}\left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right) \\
&= \frac{1}{2}r^2 f\left(br\right)\left(-2\partial_{(\mu}u_{\nu)} - \partial_r\left(-r^2 f\left(br\right)u_\mu u_\nu + r^2 P_{\mu\nu}\right)\right) \\
&\quad + \frac{1}{2}r^2\left[2\partial_{(\nu}\left(f\left(br\right)u_{\mu)}\right) - 2\partial_{(\nu}u_{\mu)}\right. \\
&\quad \left. -u^\rho\partial_\rho\left(-f\left(br\right)u_\mu u_\nu + P_{\mu\nu}\right)\right] \\
&\quad + r^2 f\left(br\right)\left(-b\partial_r\left(r^2 F\left(br\right)\right)\sigma_{\mu\nu}\right. \\
&\quad \left. -\frac{\partial_\lambda u^\lambda}{d-1}u_\mu u_\nu + \frac{1}{2}u^\lambda u_\lambda\left(u_\mu u_\nu\right)\right) \\
&\quad -\frac{\partial_\lambda u^\lambda}{d-1}r\partial_r\left(r^2 f\left(br\right)\right)u_\mu u_\nu + 2r^2\frac{\partial_\lambda u^\lambda}{d-1}P_{\mu\nu} \qquad \text{(B.13)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{r(0+1+2)}_{r\mu} &= \frac{1}{2}g^{ra}\left(g_{\mu a,r} + g_{ra,\mu} - g_{\mu r,a}\right) \\[4pt]
&= \frac{1}{2}g^{rr}\left(g_{\mu r,r} + g_{rr,\mu} - g_{\mu r,r}\right) + \frac{1}{2}g^{r\lambda}\left(g_{\mu\lambda,r} + g_{r\lambda,\mu} - g_{\mu r,\lambda}\right) \\[4pt]
&= \frac{1}{2}\left(u^\nu - \left(\mathfrak{G}^{-1}\right)^{\nu\sigma}\mathcal{V}_\sigma\right)\left(-2\partial_{[\mu}u_{\nu]} + \partial_r\left(-2u_{(\mu}\mathcal{V}_{\nu)} + \mathfrak{G}_{\mu\nu}\right)\right) \\[4pt]
&= -u_\mu\frac{\partial_\lambda u^\lambda}{d-1} - u_\mu\partial_r\left[\frac{1}{2\left(br\right)^d}\left(r^2\left(1-\left(br\right)^d\right)\right.\right. \\[6pt]
&\qquad \left.\left. -\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} - \left(br\right)^2 K_2\left(br\right)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}\right)\right] \\[6pt]
&\quad + \frac{1}{2r}u^\lambda\partial_\lambda u_\sigma P^\lambda_\mu\partial_\lambda u^\sigma - br\partial_r F\left(br\right)u^\lambda\partial_\lambda u_\sigma\sigma^\sigma_\mu \\[6pt]
&\quad - \frac{1}{r}\frac{2}{d-2}P^\rho_\mu\left(\partial_\lambda - \left(d-3\right)A_\lambda\right)\omega^\lambda_\rho \\[6pt]
&\quad + \left(\frac{2L\left(br\right)}{r\left(br\right)^{d-2}} - \frac{1}{2}\partial_r\left(\frac{2L\left(br\right)}{\left(br\right)^{d-2}}\right) + \frac{1}{r}\frac{1}{d-2}\right) \\[6pt]
&\quad \times P^\rho_\mu\left(\partial_\lambda - \left(d-1\right)A_\lambda\right)\sigma^\lambda_\rho \\[6pt]
&\quad - \frac{1}{2r}u^\lambda\partial_\lambda u^\sigma\partial_\sigma u_\mu
\end{aligned}
\tag{B.14}
$$

$$
\begin{aligned}
\Gamma^{\nu(0+1+2)}_{\nu r} &= \frac{1}{2}g^{\nu a}\left(g_{\nu a,r} + g_{ra,\nu} - g_{\nu r,a}\right) \\[4pt]
&= \frac{1}{2}g^{\nu r}\left(g_{\nu r,r} + g_{rr,\nu} - g_{\nu r,r}\right) + \frac{1}{2}g^{\nu\lambda}\left(g_{\nu\lambda,r} + g_{r\lambda,\nu} - g_{\nu r,\lambda}\right) \\[4pt]
&= \frac{\mathbf{1}}{\mathbf{2}}\left(\mathfrak{G}^{-1}\right)^{\nu\sigma}\partial_r\mathfrak{G}_{\nu\sigma} \\[4pt]
&= \frac{1}{r}\left(d-1\right) - b^2\partial_r K_1\left(br\right)\sigma_{\alpha\beta}\sigma^{\alpha\beta} \\[6pt]
&\quad + \frac{b^2}{r}P^{\alpha\beta}u^\rho\partial_\rho\sigma_{\alpha\beta}\partial_r\left(H_2\left(br\right) - H_1\left(br\right)\right) \\[6pt]
&\quad + \frac{1}{r^3}\omega_{\alpha\beta}\omega^{\alpha\beta}
\end{aligned}
\tag{B.15}
$$

$$
\begin{aligned}
\Gamma^{\nu(0+1+2)}_{\nu\mu} &= \frac{1}{2}g^{\nu a}\left(g_{\nu a,\mu} + g_{\mu a,\nu} - g_{\nu\mu,a}\right)\\[4pt]
&= \frac{1}{2}g^{\nu r}\left(g_{\nu r,\mu} + g_{\mu r,\nu} - g_{\nu\mu,r}\right) + \frac{1}{2}g^{\nu\lambda}\left(g_{\nu\lambda,\mu} + g_{\mu\lambda,\nu} - g_{\nu\mu,\lambda}\right)\\[4pt]
&= \frac{1}{2}\left(u^{\nu} - \left(\mathfrak{G}^{-1}\right)^{\nu\sigma}\mathcal{V}_{\sigma}\right)\left(-2\partial_{(\mu}u_{\nu)} - \partial_r\left(-2u_{(\mu}\mathcal{V}_{\nu)} + \mathfrak{G}_{\mu\nu}\right)\right)\\[4pt]
&\quad + \frac{1}{2}\left(\mathfrak{G}^{-1}\right)^{\nu\sigma}\partial_{\mu}\left(-2u_{(\sigma}\mathcal{V}_{\nu)} + \mathfrak{G}_{\sigma\nu}\right)\\[4pt]
&= u_{\mu}\frac{\partial_{\lambda}u^{\lambda}}{d-1} + u_{\mu}\partial_r\left[\frac{1}{2\left(br\right)^d}\left(r^2\left(1-\left(br\right)^d\right)\right.\right.\\[4pt]
&\qquad \left.\left. -\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} - \left(br\right)^2 K_2\left(br\right)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}\right)\right]\\[4pt]
&\quad -\frac{1}{2r}u^{\lambda}\partial_{\lambda}u_{\sigma}P^{\lambda}_{\mu}\partial_{\lambda}u^{\sigma} + br\partial_r F\left(br\right)u^{\lambda}\partial_{\lambda}u_{\sigma}\sigma^{\sigma}_{\mu}\\[4pt]
&\quad +\frac{1}{r}\frac{2}{d-2}P^{\rho}_{\mu}\left(\partial_{\lambda} - \left(d-3\right)A_{\lambda}\right)\omega^{\lambda}_{\rho}\\[4pt]
&\quad +\left(\frac{2L\left(br\right)}{r\left(br\right)^{d-2}} - \frac{1}{2}\partial_r\left(\frac{2L\left(br\right)}{\left(br\right)^{d-2}}\right)\right.\\[4pt]
&\qquad \left. +\frac{1}{r}\frac{1}{d-2}\right)P^{\rho}_{\mu}\left(\partial_{\lambda} - \left(d-1\right)A_{\lambda}\right)\sigma^{\lambda}_{\rho}\\[4pt]
&\quad +\frac{1}{2r}u^{\lambda}\partial_{\lambda}u^{\sigma}\partial_{\sigma}u_{\mu}\\[4pt]
&= -\Gamma^{r(0+1+2)}_{r\mu}
\end{aligned}
\tag{B.16}
$$

$$
\ell^{\mu}\Gamma^{\sigma(0+1+2)}_{r\mu}\ell_{\sigma} = 0
\tag{B.17}
$$

Appendix C

The Constarint Equations to The Third Order

We will present here a different method than what we presented in chapter 6 to find the constraint equations to the third order of the event and apparent horizons.

In order to do so, we look at equation (4.23) and use (4.20) to get the following form of the constraint equations:

$$R_{ab}m^a e^b_\mu = 0 \tag{C.1}$$

In order to check that we get the same constraint equations from the two hyper-surfaces, we calculate the difference between the constraint equations evaluated on the event horizon and the constraint equations evaluated on the apparent horizon, to the third order in derivative expansion, i.e.:

$$EH - AH \equiv R_{ab}m^a e^b_\mu \big|_{r=r_{EH}} - R_{ab}m^a e^b_\mu \big|_{r=r_{AH}} \tag{C.2}$$

The quantities will be calculated from the full second order bulk metric (3.7). Bearing in mind that the difference in m^a only starts

from the second order, and that the difference in R_{ab} and e^a_μ only starts from the third order, we can write (C.2) in the following manner:

$$
\begin{aligned}
EH - AH \;=\; & \left(R^{(3)}_{a\mu}\big|_{r=r_{EH}} - R^{(3)}_{a\mu}\big|_{r=r_{AH}} \right) m^{a(0)} \\
& + \left(m^{a(2)}\big|_{r=r_{EH}} - m^{a(2)}\big|_{r=r_{AH}} \right) R^{(1)}_{a\mu} + \\
& + \left(e^{b(3)}_\mu\big|_{r=r_{EH}} - e^{b(3)}_\mu\big|_{r=r_{AH}} \right) R^{(0)}_{ab} m^{a(0)} \\
& + \left(m^{a(3)}\big|_{r=r_{EH}} - m^{a(3)}\big|_{r=r_{AH}} \right) R^{(0)}_{a\mu} + \\
& + \left(m^{a(2)}\big|_{r=r_{EH}} - m^{a(2)}\big|_{r=r_{AH}} \right) R^{(0)}_{ab} e^{b(1)}_\mu + \\
& + \left(R^{(2)}_{ab}\big|_{r=r_{EH}} - R^{(2)}_{ab}\big|_{r=r_{AH}} \right) m^{a(0)} e^{b(1)}_\mu \\
& + \left(R^{(2)}_{a\mu}\big|_{r=r_{EH}} - R^{(2)}_{a\mu}\big|_{r=r_{AH}} \right) m^{a(1)} \qquad \text{(C.3)}
\end{aligned}
$$

where the superscript refers to the order that we are evaluating the specific quantities, i.e., if it is written $^{(1)}$ only the first derivative terms should be taken (no zero terms). The quantities in (C.3) that are outside the brackets should be evaluated at the location $r = b^{-1}$, for instance:

$$
m^{r(0)}\big|_{r=b^{-1}} = g^{rr(0)}\big|_{r=b^{-1}} = r^2 f\,(br)\big|_{r=b^{-1}} = 0 \qquad \text{(C.4)}
$$

Here we write some of the quantities that will help us to evaluate (C.3),

$$
\begin{aligned}
& R^{(0)}_{rr} = 0, \\
& m^{r(0)} = 0, \quad m^{\nu(1)} = 0, \\
& m^{\mu(2)}\big|_{r=r_{EH}} = m^{\mu(2)}\big|_{r=r_{AH}}, \\
& R^{(0)}_{r\nu} = du_\nu, \quad R^{(1)}_{r\nu} = R^{(2)}_{r\nu} = R^{(3)}_{r\nu} = 0, \\
& e^{a(1)}_{\mu EH/AH} = e^{r(1)}_{\mu EH/AH} = \partial_\mu r^{(0)}_{EH/AH} = \partial_\mu b^{-1}, \quad e^{a(3)}_\mu = e^{r(3)}_\mu = \partial_\mu r^{(2)}
\end{aligned}
\qquad \text{(C.5)}
$$

We know that only the third order terms that will be different are the one that are composed of the second order horizon location $r^{(2)}$, which is different in both hyper-surfaces. For instance, the only part that will contribute in the third order Ricci tensor, is the Ricci tensor calculated to the first order in derivatives evaluated on one of the hyper-surfaces and from this we will take the third order terms that come from $r^{(2)}_{EH/AH}$. We denote $\triangleq$ to represent the only terms that will not cancel out in the difference $m^{a(3)}\big|_{r=r_{EH}} - m^{a(3)}\big|_{r=r_{AH}}$, for example we will write,

$$
\begin{aligned}
m^{\mu(3)}\big|_{r=r_{EH/AH}} &= g^{\mu r(3)} - g^{\mu\nu(0)}\partial_\nu\left(r^{(2)}_{EH/AH}(x)\right) - g^{\mu\nu(2)}\partial_\nu\left(r^{(0)}(x)\right) \\
&\triangleq -\frac{1}{r}u^\lambda\partial_\lambda u^\mu - \frac{1}{r^2}P^{\mu\nu}\partial_\mu r^{(2)}_{EH/AH} - \frac{1}{r^2}P^{\mu\nu}\partial_\nu b^{-1} \\
&= -\frac{1}{r}u^\lambda\partial_\lambda u^\mu + \frac{1}{r^2}b^{-1}u^\lambda\partial_\lambda u^\mu - b^2 P^{\mu\nu}\partial_\mu r^{(2)}_{EH/AH} \\
&\triangleq -b^2 r^{(2)}_{EH/AH}u^\lambda\partial_\lambda u^\mu - b^2 P^{\mu\nu}\partial_\mu r^{(2)}_{EH/AH} \qquad\qquad \text{(C.6)}
\end{aligned}
$$

$$
\begin{aligned}
m^{r(3)}\big|_{r=r_{EH/AH}} &= g^{rr(3)} - g^{r\nu(0)}\partial_\nu\left(r^{(2)}_{EH/AH}(x)\right) - g^{r\nu(2)}\partial_\nu\left(r^{(0)}(x)\right) \\
&\triangleq -2r\frac{\partial_\lambda u^\lambda}{d-1} - u^\nu\partial_\nu r^{(2)}_{EH/AH} \\
&\triangleq -2r^{(2)}_{EH/AH}\frac{\partial_\lambda u^\lambda}{d-1} - u^\nu\partial_\nu r^{(2)}_{EH/AH} \qquad\qquad \text{(C.7)}
\end{aligned}
$$

where $r_{EH/AH}$ is the event or apparent horizon full location up to the second order, and $r^{(0)} = b^{-1}$.

Now we are ready to evaluate each term in (C.3):

$$\left(R_{a\mu}^{(2)} \big|_{r=r_{EH}} - R_{a\mu}^{(2)} \big|_{r=r_{AH}} \right) m^{a(1)} =$$
$$= \left(R_{r\mu}^{(2)} \big|_{r=r_{EH}} - R_{r\mu}^{(2)} \big|_{r=r_{AH}} \right) m^{r(1)} + \left(R_{\nu\mu}^{(2)} \big|_{r=r_{EH}} - R_{\nu\mu}^{(2)} \big|_{r=r_{AH}} \right) m^{\nu(1)}$$
$$= 0$$

$$\left(R_{ab}^{(2)} \big|_{r=r_{EH}} - R_{ab}^{(2)} \big|_{r=r_{AH}} \right) m^{a(0)} e_\mu^{b(1)} =$$
$$= \left(R_{rr}^{(2)} \big|_{r=r_{EH}} - R_{rr}^{(2)} \big|_{r=r_{AH}} \right) m^{r(0)} e_\mu^{r(1)} + \left(R_{\mu r}^{(2)} \big|_{r=r_{EH}} - R_{\nu r}^{(2)} \big|_{r=r_{AH}} \right) m^{\nu(0)} e_\mu^{r(1)}$$
$$= 0$$

$$\left(m^{a(2)} \big|_{r=r_{EH}} - m^{a(2)} \big|_{r=r_{AH}} \right) R_{ab}^{(0)} e_\mu^{b(1)} =$$
$$= \left(m^{r(2)} \big|_{r=r_{EH}} - m^{r(2)} \big|_{r=r_{AH}} \right) R_{rr}^{(0)} e_\mu^{r(1)} + \left(m^{\nu(2)} \big|_{r=r_{EH}} - m^{\nu(2)} \big|_{r=r_{AH}} \right) R_{\nu r}^{(0)} e_\mu^{r(1)}$$
$$= 0$$

$$\left(m^{a(3)} \big|_{r=r_{EH}} - m^{a(3)} \big|_{r=r_{AH}} \right) R_{a\mu}^{(0)} =$$
$$= \left(m^{r(3)} \big|_{r=r_{EH}} - m^{r(3)} \big|_{r=r_{AH}} \right) R_{r\mu}^{(0)} + \left(m^{\nu(3)} \big|_{r=r_{EH}} - m^{\nu(3)} \big|_{r=r_{AH}} \right) R_{\nu\mu}^{(0)}$$
$$= -d \left(r^{(2)} \big|_{r=r_{EH}} - r^{(2)} \big|_{r=r_{AH}} \right) 2 \frac{\partial_\lambda u^\lambda}{d-1} u_\mu$$
$$+ d b^{-2} P_{\mu\nu} \left(r^{(2)} \big|_{r=r_{EH}} - r^{(2)} \big|_{r=r_{AH}} \right) b^2 u^\lambda \partial_\lambda u^\nu$$
$$- d u^\nu u_\mu \left(\partial_\nu \left(r^{(2)}(x) \right) \big|_{r=r_{EH}} - \partial_\nu \left(r^{(2)}(x) \right) \big|_{r=r_{AH}} \right)$$
$$+ d P_\mu^\sigma \left(\partial_\sigma \left(r^{(2)}(x) \right) \big|_{r=r_{EH}} - \partial_\sigma \left(r^{(2)}(x) \right) \big|_{r=r_{AH}} \right)$$
$$= -d \left(r^{(2)} \big|_{r=r_{EH}} - r^{(2)} \big|_{r=r_{AH}} \right) \left(2 \frac{\partial_\lambda u^\lambda}{d-1} u_\mu - u^\lambda \partial_\lambda u_\nu \right)$$
$$+ d \left(\partial_\mu \left(r^{(2)}(x) \right) \big|_{r=r_{EH}} - \partial_\mu \left(r^{(2)}(x) \right) \big|_{r=r_{AH}} \right)$$

$$\left(e_\mu^{b(3)} \big|_{r=r_{EH}} - e_\mu^{b(3)} \big|_{r=r_{AH}} \right) R_{ab}^{(0)} m^{a(0)} =$$
$$= \left(e_\mu^{r(3)} \big|_{r=r_{EH}} - e_\mu^{r(3)} \big|_{r=r_{AH}} \right) R_{rr}^{(0)} m^{r(0)} + \left(e_\mu^{r(3)} \big|_{r=r_{EH}} - e_\mu^{r(3)} \big|_{r=r_{AH}} \right) R_{r\nu}^{(0)} m^{\nu(0)}$$
$$= -d \left(\partial_\mu \left(r^{(2)}(x) \right) \big|_{r=r_{EH}} - \partial_\mu \left(r^{(2)}(x) \right) \big|_{r=r_{AH}} \right)$$

$$\left(m^{a(2)} \big|_{r=r_{EH}} - m^{a(2)} \big|_{r=r_{AH}} \right) R_{a\mu}^{(1)} =$$
$$= \left(m^{\nu(2)} \big|_{r=r_{EH}} - m^{\nu(2)} \big|_{r=r_{AH}} \right) R_{\nu\mu}^{(1)} + \left(m^{r(2)} \big|_{r=r_{EH}} - m^{r(2)} \big|_{r=r_{AH}} \right) R_{r\mu}^{(1)} = 0$$

$$\left(R_{a\mu}^{(3)} \big|_{r=r_{EH}} - R_{a\mu}^{(3)} \big|_{r=r_{AH}} \right) m^{a(0)} =$$
$$= \left(R_{r\mu}^{(3)} \big|_{r=r_{EH}} - R_{r\mu}^{(3)} \big|_{r=r_{AH}} \right) m^{r(0)} + \left(R_{\nu\mu}^{(3)} \big|_{r=r_{EH}} - R_{\nu\mu}^{(3)} \big|_{r=r_{AH}} \right) m^{\nu(0)}$$
$$= \left(R_{\nu\mu}^{(3)} \big|_{r=r_{EH}} - R_{\nu\mu}^{(3)} \big|_{r=r_{AH}} \right) u^\nu$$
$$= d \left(r^{(2)} \big|_{r=r_{EH}} - r^{(2)} \big|_{r=r_{AH}} \right) \left(2 \frac{\partial_\lambda u^\lambda}{d-1} u_\mu - u^\lambda \partial_\lambda u_\nu \right)$$

We seek the third order constraint equations. This is the reason that in our calculation we used the previous constraint equations (5.16), (5.17) that are of the second order. Assembling all of the above terms and inserting them into (C.3), reveals that our statement is correct, for we found:

$$EH - AH \equiv R_{ab} m^a e_\mu^b \big|_{r=r_{EH}} - R_{ab} m^a e_\mu^b \big|_{r=r_{AH}} = 0 \quad (C.8)$$

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